

Problem 1.

Let the random variable Y be $Y = \sin(5X + \frac{\pi}{4})$, where X is an exponentially distributed random variable with mean $\frac{1}{\lambda}$, i.e.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0, & x < 0, \end{cases}.$$

What is the PDF f_Y of the random variable Y ?

Problem 2. Let X and Y be independent and identically distributed random variables with the following PDF

$$f_X(x) = \begin{cases} \frac{1}{x^2} & x \geq 1, \\ 0, & x < 1, \end{cases}.$$

- (a) Let the random variable Z be $Z = XY$. What is the PDF of Z ?
- (b) Let the random variable W be $W = X/Y$. What is the PDF of W ?
- (c) Let the random variable A be $A = \min\{X, Y\}$. What is the PDF of A ?
- (d) Let the random variable B be $B = \max\{X, Y\}$. What is the PDF of B ?

Solution of Problem 1. We can solve this problem either starting from the first principles and by using the CDF or by plugging in the formula for the derived distributions.

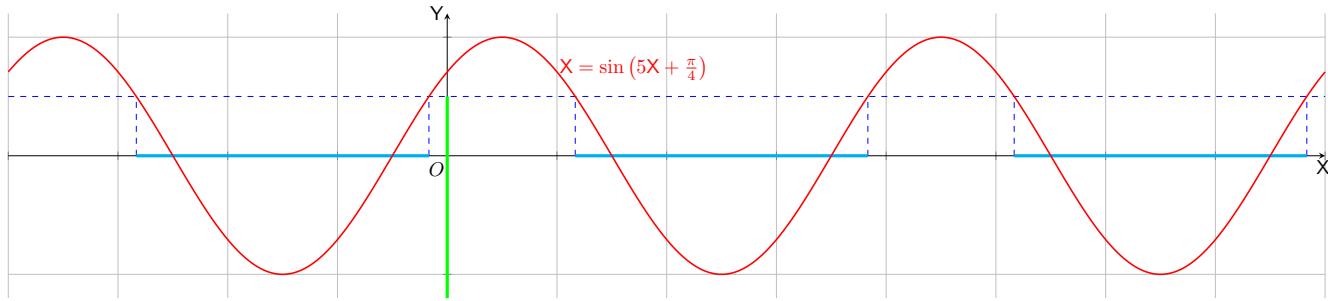


Figure 1.

a) Let us first present the solution via CDF's. Since $Y = \sin(5X + \frac{\pi}{4})$, we can write the event $\{Y \leq y\}$ in terms of X as

$$\begin{aligned} \{Y \leq y\} &= \{\sin(5X + \frac{\pi}{4}) \leq y\} \\ &= \bigcup_{k \in \mathbb{Z}} \left\{ 5X + \frac{\pi}{4} \in [-\arcsin(y) - \pi + 2\pi k, \arcsin(y) + 2\pi k] \right\} \\ &= \bigcup_{k \in \mathbb{Z}} \left\{ X \in \left[-\frac{\arcsin(y)}{5} - \frac{\pi}{4} + \frac{2\pi k}{5}, \frac{\arcsin(y)}{5} - \frac{\pi}{20} + \frac{2\pi k}{5} \right] \right\}. \end{aligned}$$

Thus using the PDF of X we get

$$\begin{aligned} F_Y(y) &= \sum_{k=-\infty}^{\infty} \int_{-\frac{\arcsin(y)}{5} - \frac{\pi}{4} + \frac{2\pi k}{5}}^{\frac{\arcsin(y)}{5} - \frac{\pi}{20} + \frac{2\pi k}{5}} f_X(x) dx \\ &= \int_0^{\frac{\arcsin(y)}{5} - \frac{\pi}{20}} f_X(x) dx + \sum_{k=1}^{\infty} \int_{-\frac{\arcsin(y)}{5} - \frac{\pi}{4} + \frac{2\pi k}{5}}^{\frac{\arcsin(y)}{5} - \frac{\pi}{20} + \frac{2\pi k}{5}} \lambda e^{-\lambda x} dx \quad \forall y \in [-1, 1]. \end{aligned}$$

On the other hand

$$\int_0^{\frac{\arcsin(y)}{5} - \frac{\pi}{20}} f_X(x) dx = \begin{cases} \int_0^{\frac{\arcsin(y)}{5} - \frac{\pi}{20}} \lambda e^{-\lambda x} dx & \arcsin(y) \geq \frac{\pi}{4} \\ 0 & \arcsin(y) < \frac{\pi}{4} \end{cases}$$

Thus

$$F_Y(y) = \begin{cases} 1 & y \in (1, \infty) \\ \int_0^{\frac{\arcsin(y)}{5} - \frac{\pi}{20}} \lambda e^{-\lambda x} dx + \sum_{k=1}^{\infty} \int_{-\frac{\arcsin(y)}{5} - \frac{\pi}{4} + \frac{2\pi k}{5}}^{\frac{\arcsin(y)}{5} - \frac{\pi}{20} + \frac{2\pi k}{5}} \lambda e^{-\lambda x} dx & y \in [\frac{1}{\sqrt{2}}, 1] \\ \sum_{k=1}^{\infty} \int_{-\frac{\arcsin(y)}{5} - \frac{\pi}{4} + \frac{2\pi k}{5}}^{\frac{\arcsin(y)}{5} - \frac{\pi}{20} + \frac{2\pi k}{5}} \lambda e^{-\lambda x} dx & y \in [-1, \frac{1}{\sqrt{2}}) \\ 0 & y \in (-\infty, 1) \end{cases}$$

Using $\frac{d}{dy} F_Y = f_Y$, $\frac{d}{dy} \arcsin(y) = \frac{1}{\sqrt{1-y^2}}$, and $\frac{d}{dy} \int_a^b f(x) dx = f(b) \frac{d}{dy} b - f(a) \frac{d}{dy} a$, we get

$$\begin{aligned} f_Y(y) &= \frac{\lambda}{5\sqrt{1-y^2}} \begin{cases} e^{-\lambda \frac{\arcsin(y)}{5} + \lambda \frac{\pi}{20}} \sum_{k=0}^{\infty} e^{-\lambda \frac{2\pi k}{5}} + e^{\lambda \frac{\arcsin(y)}{5} + \lambda \frac{\pi}{4}} \sum_{k=1}^{\infty} e^{-\lambda \frac{2\pi k}{5}} & y \in [\frac{1}{\sqrt{2}}, 1] \\ e^{-\lambda \frac{\arcsin(y)}{5} + \lambda \frac{\pi}{20}} \sum_{k=1}^{\infty} e^{-\lambda \frac{2\pi k}{5}} + e^{\lambda \frac{\arcsin(y)}{5} + \lambda \frac{\pi}{4}} \sum_{k=1}^{\infty} e^{-\lambda \frac{2\pi k}{5}} & y \in [-1, \frac{1}{\sqrt{2}}) \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{\lambda}{5\sqrt{1-y^2}} \frac{1}{1-e^{-\lambda \frac{2\pi}{5}}} \begin{cases} e^{-\lambda \frac{\arcsin(y)-\pi/4}{5}} + e^{\lambda \frac{\arcsin(y)-3\pi/4}{5}} & y \in [\frac{1}{\sqrt{2}}, 1] \\ e^{-\lambda \frac{\arcsin(y)+7\pi/4}{5}} + e^{\lambda \frac{\arcsin(y)-3\pi/4}{5}} & y \in [-1, \frac{1}{\sqrt{2}}) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

b) Now let us invoke the formula for derived distributions to obtain f_Y from f_X :

$$\begin{aligned}
 f_Y(y) &= \sum_{x:y=\sin(5x+\pi/4)} \frac{1}{\left| \frac{dy}{dx} \right|_{z=x}} f_X(x) \\
 &= \sum_{x:y=\sin(5x+\pi/4)} \frac{1}{5|\cos(5z+\pi/4)|_{z=x}} f_X(x) \\
 &= \sum_{x:y=\sin(5x+\pi/4)} \frac{1}{5\sqrt{1-y^2}} f_X(x) \\
 &= \begin{cases} \sum_{k \in \mathbb{Z}} \frac{1}{5\sqrt{1-y^2}} f_X\left(\frac{\arcsin(y)-\pi/4+2\pi k}{5}\right) + \sum_{k \in \mathbb{Z}} \frac{1}{5\sqrt{1-y^2}} f_X\left(\frac{\pi-\arcsin(y)-\pi/4+2\pi k}{5}\right) & y \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \\
 &= \frac{\lambda}{5\sqrt{1-y^2}} \begin{cases} e^{-\lambda \frac{\arcsin(y)-\pi/4}{5}} \sum_{k=0}^{\infty} e^{-\lambda \frac{2\pi k}{5}} + e^{\lambda \frac{\arcsin(y)-3\pi/4}{5}} \sum_{k=0}^{\infty} e^{-\lambda \frac{2\pi k}{5}} & y \in [\frac{1}{\sqrt{2}}, 1] \\ e^{-\lambda \frac{\arcsin(y)-\pi/4}{5}} \sum_{k=1}^{\infty} e^{-\lambda \frac{2\pi k}{5}} + e^{\lambda \frac{\arcsin(y)-3\pi/4}{5}} \sum_{k=0}^{\infty} e^{-\lambda \frac{2\pi k}{5}} & y \in [-1, \frac{1}{\sqrt{2}}) \\ 0 & \text{otherwise} \end{cases} \quad (1) \\
 &= \frac{\lambda}{5\sqrt{1-y^2}} \frac{1}{1-e^{-\lambda \frac{2\pi}{5}}} \begin{cases} e^{-\lambda \frac{\arcsin(y)-\pi/4}{5}} + e^{\lambda \frac{\arcsin(y)-3\pi/4}{5}} & y \in [\frac{1}{\sqrt{2}}, 1] \\ e^{-\lambda \frac{\arcsin(y)+7\pi/4}{5}} + e^{\lambda \frac{\arcsin(y)-3\pi/4}{5}} & y \in [-1, \frac{1}{\sqrt{2}}) \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

As expected both expected both calculations leads to exactly the same result. \square

Solution of Problem 2.

- (a) Note that $\ln X$ and $\ln Y$ are independent random variables because X and Y are independent random variables. Thus $\ln Z = \ln X + \ln Y$ imply

$$f_{\ln Z}(s) = \int f_{\ln X}(\tau) f_{\ln Y}(s - \tau) d\tau \quad (2)$$

On the other hand

$$\begin{aligned} f_{\ln X}(\tau) &= \frac{1}{1/e^\tau} f_X(e^\tau) \\ &= \begin{cases} e^{-\tau}, & \tau \geq 0, \\ 0, & \tau < 0, \end{cases} \end{aligned} \quad (3)$$

Note that (2), (3), and $f_X(\tau) = f_Y(\tau)$ imply

$$\begin{aligned} f_{\ln Z}(s) &= \begin{cases} \int_0^s e^{-\tau} e^{-(s-\tau)} d\tau & s \geq 0 \\ 0 & s < 0 \end{cases} \\ &= \begin{cases} se^{-s} & s \geq 0 \\ 0 & s < 0 \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} f_Z(\tau) &= \frac{1}{\tau} f_{\ln Z}(\ln \tau) \\ &= \begin{cases} \frac{\ln \tau}{\tau^2} & \tau \geq 1 \\ 0 & \tau < 1 \end{cases} \end{aligned} \quad (4)$$

- (b) Note that $\ln X$ and $-\ln Y$ are independent random variables because X and Y are independent random variables. Thus $\ln W = \ln X + (-\ln Y)$ imply

$$f_{\ln W}(s) = \int f_{\ln X}(\tau) f_{-\ln Y}(s - \tau) d\tau \quad (5)$$

On the other hand $f_X(\tau) = f_Y(\tau)$ imply

$$\begin{aligned} f_{-\ln Y}(\tau) &= f_{\ln X}(-\tau) \\ &= \begin{cases} e^\tau, & \tau \leq 0, \\ 0, & \tau > 0, \end{cases} \end{aligned} \quad (6)$$

Note that (3), (5), and (6) imply

$$\begin{aligned} f_{\ln W}(s) &= \begin{cases} e^s \int_s^\infty e^{-2\tau} d\tau & s \geq 0 \\ e^s \int_0^\infty e^{-2\tau} d\tau & s < 0 \end{cases} \\ &= \frac{1}{2} e^{-|s|} \end{aligned}$$

Thus for $\tau > 0$ we have

$$\begin{aligned} f_W(\tau) &= \frac{1}{\tau} f_{\ln W}(\ln \tau) \\ &= \frac{1}{\tau} \frac{1}{2} e^{-|\ln \tau|} \end{aligned}$$

On the other han $f_W(\tau) = 0$ for $\tau \leq 0$ because the range of the function $g(\tau) = e^\tau$ is the set of positive real numbers. Hence

$$f_W(\tau) = \begin{cases} \frac{1}{2} \min\{1, \frac{1}{\tau^2}\} & \tau > 0 \\ 0 & \tau \leq 0 \end{cases}. \quad (7)$$

- (c) Note that $A > s$ iff $X > s$ and $Y > s$. Thus as

$$\begin{aligned} \mathbf{P}(A > s) &= \mathbf{P}(X > s \text{ and } Y > s) \\ &= \mathbf{P}(X > s) \mathbf{P}(Y > s) && \text{because } X \text{ and } Y \text{ are independent.} \\ &= \begin{cases} \left(\int_s^\infty \frac{1}{\tau^2} d\tau\right)^2 & s > 1 \\ 1 & s \leq 1 \end{cases} \\ &= \frac{1}{\max\{1,s\}^2} \end{aligned}$$

Then using $F_A(s) = 1 - \mathbf{P}(A > s)$ and $\frac{d}{ds}F_A(s) = f_A(s)$ we get,

$$f_A(\tau) = \begin{cases} 2\tau^{-3} & \tau \geq 1 \\ 0 & \tau < 1 \end{cases}$$

(d) Note that $B \leq s$ iff $X \leq s$ and $Y \leq s$. Thus as

$$\begin{aligned} F_B(s) &= \mathbf{P}(B \leq s) \\ &= \mathbf{P}(X \leq s \text{ and } Y \leq s) \\ &= \mathbf{P}(X \leq s) \mathbf{P}(Y \leq s) && \text{because } X \text{ and } Y \text{ are independent.} \\ &= \begin{cases} \left(\int_1^s \frac{1}{\tau^2} d\tau\right)^2 & s > 1 \\ 0 & s \leq 1 \end{cases} \\ &= \begin{cases} \left(1 - \frac{1}{s}\right)^2 & s > 1 \\ 0 & s \leq 1 \end{cases} \end{aligned}$$

Then using $\frac{d}{ds}F_B(s) = f_B(s)$ we get,

$$f_B(\tau) = \begin{cases} 2\frac{\tau-1}{\tau^3} & \tau \geq 1 \\ 0 & \tau < 1 \end{cases}$$

□