Problem 1.
Let the PDF of the random variable X be

$$f_{\mathsf{X}}(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1, \\ 0, & x < 1, \end{cases}$$

and the conditional PDF $f_{\mathsf{Y}|\mathsf{X}}(\cdot|x)$ of the random variable Y given X be

$$f_{\mathsf{Y}|\mathsf{X}}(y|x) = \begin{cases} \gamma(x)e^{-y/x}, & \text{if } y \geq x, \\ 0, & \text{else}, \end{cases}$$

for all $x \geq 1$, where $\gamma(x)$ is a function of x to be determined.

(a) Determine $\gamma(x)$ for all $x \geq 1$.
(b) Determine the conditional mean of Y given $\mathsf{X} = x$, i.e. $\mathbf{E}[\mathsf{Y}|\,\mathsf{X} = x]$, in terms of x.
(c) Determine the mean of Y, i.e. $\mathbf{E}[\mathsf{Y}]$.
(d) Determine conditional PDF of X given Y, i.e. $f_{\mathsf{X}|\mathsf{Y}}$.

Problem 2.

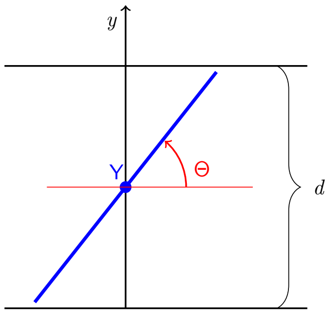

Figure 1.

A length 2ℓ needle is dropped between two horizontal lines that are separated by a distance d in such a way that the joint probability density function of the distance of needle's midpoint from the lower line, i.e. the continuous random variable Y, and the angle needle makes with horizontal lines, i.e. the continuous random variable Θ, are uniformly distributed as follows

$$f_{\mathsf{Y},\Theta}(y,\theta) = \begin{cases} \frac{1}{d\pi}, & \text{if } y \in [0,d], \text{ and } \theta \in [0,\pi] \ , \\ 0, & \text{otherwise}, \end{cases},$$

where $\frac{d}{2} < \ell < d$. The needle may cross $y = 0$ and $y = d$ lines and the random variable N is equal to the total number of lines needle crosses.

(a) What is the conditional PMF $p_{\mathsf{N}|\mathsf{Y}}$ of N given Y ?
(b) What is the conditional CDF $f_{\mathsf{Y}|\mathsf{N}}$ of Y given N?

Solution of Problem 1.

(a) The conditional PDF $f_{\mathsf{Y}|\mathsf{X}}(y|x)$ has to be a PDF for all $x \geq 1$:

$$1 = \int_{-\infty}^{\infty} f_{\mathsf{Y}|\mathsf{X}}(y|x) \qquad = \gamma(x) \int_{x}^{\infty} e^{-\frac{y}{x}} \mathrm{d}y \qquad = x\gamma(x) \int_{1}^{\infty} e^{-\tau} \mathrm{d}\tau \qquad = x\gamma(x) e^{-1}.$$

Hence $\gamma(x) = \frac{e}{x}$ and

$$f_{\mathsf{Y}|\mathsf{X}}(y|x) = \begin{cases} \frac{1}{x} e^{-\frac{y-x}{x}}, & \text{if } y \geq x, \\ 0, & \text{if } y < x, \end{cases} \tag{1}$$

for all $x \geq 1$,

(b) Given $\mathsf{X} = x$, the random variable Y is equal to the sum of x and an exponentially distributed random variable with mean x, see Problem 35-(a) and its solution in the textbook. Thus $\mathbf{E}[\mathsf{Y}|\,\mathsf{X} = x] = 2x$. This can be confirmed using $\mathbf{E}[\mathsf{Y}|\,\mathsf{X} = x] = \int_{-\infty}^{\infty} y f_{\mathsf{Y}|\mathsf{X}}(y|x)\,\mathrm{d}y$ and the expression we obtained for $f_{\mathsf{Y}|\mathsf{X}}(y|x)$, by applying integration by parts:

$$\mathbf{E}[\mathsf{Y}|\,\mathsf{X} = x] = \int_{x}^{\infty} y \tfrac{e}{x} e^{-y/x} \mathrm{d}y \quad = ex \int_{1}^{\infty} \tau e^{-\tau} \mathrm{d}\tau \quad = ex \left[-\tau e^{-\tau}\Big|_{1}^{\infty} + \int_{1}^{\infty} e^{-\tau} \mathrm{d}\tau \right] \quad = ex \left[e^{-1} + \left(-e^{-\tau}\right)\Big|_{1}^{\infty} \right]$$
$$= 2x$$

(c) We can use the law of iterated expectations, i.e. $\mathbf{E}[\mathsf{Y}] = \mathbf{E}[\mathbf{E}[\mathsf{Y}|\,\mathsf{X}]]$.

$$\mathbf{E}[\mathsf{Y}] = \mathbf{E}[2\mathsf{X}] \qquad = 2\int_{-\infty}^{\infty} x f_{\mathsf{X}}(x)\,\mathrm{d}x \qquad = 2\int_{1}^{\infty} 2x^{-2} \mathrm{d}x \qquad = 2\left(-2x^{-1}\right)\Big|_{1}^{\infty} \qquad = 4.$$

Remark 1. We need not to calculate the PDF of Y, in order to calculate $\mathbf{E}[\mathsf{Y}]$.

(d) Since both f_{X} and $f_{\mathsf{Y}|\mathsf{X}}$ are given we can first calculate f_{Y} using f_{X} and $f_{\mathsf{Y}|\mathsf{X}}$ and then invoke Bayes' rule: $f_{\mathsf{X}|\mathsf{Y}} = \frac{f_{\mathsf{Y}|\mathsf{X}} f_{\mathsf{X}}}{f_{\mathsf{Y}}}$.

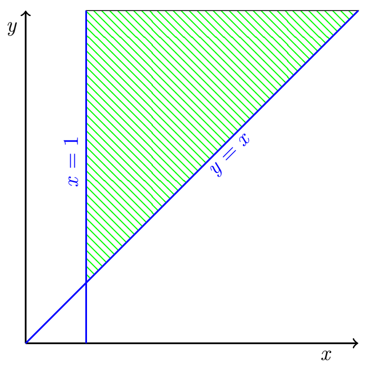

Figure 2.

We calculate f_{Y} using $f_{\mathsf{Y}}(y) = \int f_{\mathsf{X},\mathsf{Y}}(x,y)\,\mathrm{d}y$ where $f_{\mathsf{X},\mathsf{Y}}(x,y) = f_{\mathsf{Y}|\mathsf{X}}(y|x)\, f_{\mathsf{X}}(x)$. Note that $f_{\mathsf{X},\mathsf{Y}}(x,y) > 0$ iff $x \geq 1$ and $x \leq y$, this is the shaded region in Figure 2. Thus

$$f_{\mathsf{Y}}(y) = \begin{cases} 2e \int_{1}^{y} x^{-4} e^{-\frac{y}{x}} \mathrm{d}x & y \geq 1 \\ 0 & y < 1 \end{cases}$$

$$\int_{1}^{y} x^{-4} e^{-\frac{y}{x}} \mathrm{d}x = -y^{-3} \int_{y}^{1} e^{-\tau} \tau^2 \mathrm{d}\tau \qquad \text{for } x = \tfrac{y}{\tau}$$
$$= y^{-3} \left[\tau^2 e^{-\tau} + 2\tau e^{-\tau} + 2e^{-\tau}\right]\Big|_{y}^{1}$$
$$= y^{-3} \left[5e^{-1} - (y^2 + 2y + 2)e^{-y}\right].$$

Since $f_{\mathsf{Y}}(y)$ is positive only for $y > 1$, the conditional PDF $f_{\mathsf{X}|\mathsf{Y}}(x|y)$ is defined only for $y > 1$.

$$f_{\mathsf{X}|\mathsf{Y}}(x|y) = \frac{f_{\mathsf{Y}|\mathsf{X}}(y|x) f_{\mathsf{X}}(x)}{f_{\mathsf{Y}}(y)}$$
$$= \begin{cases} \frac{y^3 x^{-4}}{5e^{-1} - (y^2+2y+2)e^{-y}} e^{-\frac{y}{x}}, & \text{if } x \in [1, y], \\ 0, & \text{if } x \notin [1, y], \end{cases}$$

$\square$

Solution of Problem 2. ["iff"="if and only if"]

(a) Note that the value of N can be calculate if Y and Θ are known. In particular,

- The needle crosses $y = 0$ line iff $\mathsf{Y} \leq \ell \sin\Theta$.
- The needle crosses $y = d$ line iff $d - \mathsf{Y} \leq \ell \sin\Theta$.

Since $\Theta \in [0, \pi]$ we can rewrite these conditions as

- The needle crosses $y = 0$ line iff $\Theta \in \left[\arcsin\left(\frac{\mathsf{Y}}{\ell}\right), \pi - \arcsin\left(\frac{\mathsf{Y}}{\ell}\right)\right]$.
- The needle crosses $y = d$ line iff $\Theta \in \left[\arcsin\left(\frac{d-\mathsf{Y}}{\ell}\right), \pi - \arcsin\left(\frac{d-\mathsf{Y}}{\ell}\right)\right]$.

Note that the problem has a symmetry with respect to $y = \frac{d}{2}$ line. We confine our discussion to $\mathsf{Y} \in [0, \frac{d}{2}]$ case first. Depending on the region Y belongs there two different possibilities for the possible crossings as a function of Θ.

- If $\mathsf{Y} \in [0, d-\ell)$ then needle cannot cross $y = d$ line but it cannot cross $y = 0$ line. Thus
  - $\mathsf{N} = 0$ iff the needle does not cross $y = 0$ line:
  $$\mathsf{N} = 0 \text{ iff } \Theta \in \left\{\left[0, \arcsin\left(\tfrac{\mathsf{Y}}{\ell}\right)\right) \bigcup \left(\pi - \arcsin\left(\tfrac{\mathsf{Y}}{\ell}\right), \pi\right]\right\}.$$
  - $\mathsf{N} = 1$ iff the needle crosses $y = 0$ line:
  $$\mathsf{N} = 1 \text{ iff } \Theta \in \left[\arcsin\left(\tfrac{\mathsf{Y}}{\ell}\right), \pi - \arcsin\left(\tfrac{\mathsf{Y}}{\ell}\right)\right].$$

  Thus
  $$p_{\mathsf{N}|\mathsf{Y}}(n|y) = \begin{cases} \frac{2}{\pi}\arcsin\left(\frac{y}{\ell}\right) & n = 0 \\ 1 - \frac{2}{\pi}\arcsin\left(\frac{y}{\ell}\right) & n = 1 \end{cases}. \tag{2}$$

- If $\mathsf{Y} \in [d-\ell, \frac{d}{2}]$, then the needle can cross both $y = 0$ and $y = d$ lines. However, $y = d$ line cannot be crossed without $y = 0$ line being crossed for $\mathsf{Y} \in [d-\ell, \frac{d}{2}]$ case. This is also implied by $\arcsin\left(\frac{\mathsf{Y}}{\ell}\right) \leq \arcsin\left(\frac{d-\mathsf{Y}}{\ell}\right)$, which holds for $\mathsf{Y} \in [d-\ell, \frac{d}{2}]$ case.
  - $\mathsf{N} = 0$ iff the needle does not cross either $y = 0$ line or $y = d$ line. This is equivalent to needle not crossing $y = 0$:
  $$\mathsf{N} = 0 \text{ iff } \Theta \in \left\{\left[0, \arcsin\left(\tfrac{\mathsf{Y}}{\ell}\right)\right) \bigcup \left(\pi - \arcsin\left(\tfrac{\mathsf{Y}}{\ell}\right), \pi\right]\right\}.$$
  - $\mathsf{N} = 1$ iff the needle crosses $y = 0$ line but not $y = d$ line:
  $$\mathsf{N} = 1 \text{ iff } \Theta \in \left\{\left[\arcsin\left(\tfrac{\mathsf{Y}}{\ell}\right), \arcsin\left(\tfrac{d-\mathsf{Y}}{\ell}\right)\right) \bigcup \left(\pi - \arcsin\left(\tfrac{d-\mathsf{Y}}{\ell}\right), \pi - \arcsin\left(\tfrac{\mathsf{Y}}{\ell}\right)\right]\right\}.$$
  - $\mathsf{N} = 2$ iff the needle crosses both $y = 0$ and $y = d$ lines:
  $$\mathsf{N} = 2 \text{ iff } \Theta \in \left[\arcsin\left(\tfrac{d-\mathsf{Y}}{\ell}\right), \pi - \arcsin\left(\tfrac{d-\mathsf{Y}}{\ell}\right)\right].$$

  Thus
  $$p_{\mathsf{N}|\mathsf{Y}}(n|y) = \begin{cases} \frac{2}{\pi}\arcsin\left(\frac{y}{\ell}\right) & n = 0 \\ \frac{2}{\pi}\left[\arcsin\left(\frac{d-y}{\ell}\right) - \arcsin\left(\frac{y}{\ell}\right)\right] & n = 1 \\ 1 - \frac{2}{\pi}\arcsin\left(\frac{d-y}{\ell}\right) & n = 2 \end{cases}. \tag{3}$$

Using (2) and (3) together with the symmetry of the problem we get the following conditional PMF:

- If $y \in [0, d-\ell) \cup (\ell, d]$
$$p_{\mathsf{N}|\mathsf{Y}}(n|y) = \begin{cases} \frac{2}{\pi}\arcsin\left(\frac{\min\{y, d-y\}}{\ell}\right) & n = 0 \\ 1 - \frac{2}{\pi}\arcsin\left(\frac{\min\{y, d-y\}}{\ell}\right) & n = 1 \end{cases} \tag{4}$$
- If $y \in [d-\ell, \ell]$
$$p_{\mathsf{N}|\mathsf{Y}}(n|y) = \begin{cases} \frac{2}{\pi}\arcsin\left(\frac{\min\{y, d-y\}}{\ell}\right) & n = 0 \\ \frac{2}{\pi}\left|\arcsin\left(\frac{d-y}{\ell}\right) - \arcsin\left(\frac{y}{\ell}\right)\right| & n = 1 \\ 1 - \frac{2}{\pi}\arcsin\left(\frac{\max\{y, d-y\}}{\ell}\right) & n = 2 \end{cases} \tag{5}$$

(b) We will calculate the conditional PDF $f_{\mathsf{Y|N}}$ using the Baye's rule $f_{\mathsf{Y|N}} = \frac{p_{\mathsf{N|Y}} f_{\mathsf{Y}}}{p_{\mathsf{N}}}$. To do that we will first calculate p_{N} using $p_{\mathsf{N}}(n) = \int p_{\mathsf{N|Y}}(n|y)\, f_{\mathsf{Y}}(y)\, \mathrm{d}y$. Thus

$$\begin{aligned} p_{\mathsf{N}}(0) &= 2\int_0^{\frac{d}{2}} \tfrac{2}{\pi} \arcsin\left(\tfrac{y}{\ell}\right) \tfrac{1}{d} \mathrm{d}y \\ &= \tfrac{4\ell}{d\pi} \int_0^{\frac{d}{2\ell}} \arcsin(\tau)\, \mathrm{d}\tau \\ &= \tfrac{4\ell}{d\pi} \left[\tau \arcsin(\tau) + \sqrt{1-\tau^2}\right]\Big|_0^{\frac{d}{2\ell}} \\ &= \tfrac{4\ell}{d\pi} \left[\tfrac{d}{2\ell} \arcsin(\tfrac{d}{2\ell}) + \sqrt{1-(\tfrac{d}{2\ell})^2} - 1\right]. \end{aligned} \tag{6}$$

$$\begin{aligned} p_{\mathsf{N}}(2) &= 2\int_{d-\ell}^{\frac{d}{2}} \left[1 - \tfrac{2}{\pi} \arcsin\left(\tfrac{d-y}{\ell}\right)\right] \tfrac{1}{d} \mathrm{d}y \\ &= \tfrac{2\ell-d}{d} + \tfrac{4\ell}{d\pi} \int_1^{\frac{d}{2\ell}} \arcsin(\tau)\, \mathrm{d}\tau \\ &= \tfrac{2\ell-d}{d} + \tfrac{4\ell}{d\pi} \left[\tau \arcsin(\tau) + \sqrt{1-\tau^2}\right]\Big|_1^{\frac{d}{2\ell}} \\ &= -1 + \tfrac{4\ell}{d\pi} \left[\tfrac{d}{2\ell} \arcsin(\tfrac{d}{2\ell}) + \sqrt{1-(\tfrac{d}{2\ell})^2}\right]. \end{aligned} \tag{7}$$

We can calculate $p_{\mathsf{N}}(1)$ using $p_{\mathsf{N}}(1) = 1 - p_{\mathsf{N}}(0) - p_{\mathsf{N}}(2)$ together with (6) and (7):

$$p_{\mathsf{N}}(1) = 2 + \tfrac{4\ell}{d\pi} \left[1 - 2\left(\tfrac{d}{2\ell} \arcsin(\tfrac{d}{2\ell}) + \sqrt{1-(\tfrac{d}{2\ell})^2}\right)\right]. \tag{8}$$

Using (4), (5), and (6) we get

$$f_{\mathsf{Y|N}}(y|0) = \begin{cases} \frac{\arcsin\left(\frac{\min\{y,d-y\}}{\ell}\right)}{\arcsin(\frac{d}{2\ell}) + \sqrt{(\frac{2\ell}{d})^2 - 1} - \frac{2\ell}{d}}. & y \in [0, d] \\ 0 & y \notin [0, d] \end{cases} \tag{9}$$

Using (4), (5), and (8) we get

$$f_{\mathsf{Y|N}}(y|1) = \begin{cases} \frac{\frac{\pi}{2} - \arcsin\left(\frac{\min\{y,d-y\}}{\ell}\right)}{\pi + \left[\frac{2\ell}{d} - 2\left(\arcsin(\frac{d}{2\ell}) + \sqrt{(\frac{2\ell}{d})^2 - 1}\right)\right]} & y \in [0, d-\ell) \cup (\ell, d] \\ \frac{\left|\arcsin\left(\frac{d-y}{\ell}\right) - \arcsin\left(\frac{y}{\ell}\right)\right|}{\pi + \left[\frac{2\ell}{d} - 2\left(\arcsin(\frac{d}{2\ell}) + \sqrt{(\frac{2\ell}{d})^2 - 1}\right)\right]} & y \in [d-\ell, \ell] \\ 0 & y \notin [0, d] \end{cases}. \tag{10}$$

Using (4), (5), and (7) we get

$$f_{\mathsf{Y|N}}(y|2) = \begin{cases} \frac{\frac{\pi}{2} - \arcsin\left(\frac{\max\{d-y,y\}}{\ell}\right)}{\arcsin(\frac{d}{2\ell}) + \sqrt{(\frac{2\ell}{d})^2 - 1} - \frac{\pi}{2}} & y \in [d-\ell, \ell] \\ 0 & y \notin [d-\ell, \ell] \end{cases}. \tag{11}$$

□