I. Recommended Questions and Solutions of Select Questions

- **Recommended Questions For Section 3.3:** 11, 12, 13
- **Recommended Questions For Section 3.4:** 15, 16
  Solutions of Problems 15 and 16 are presented in the following.
- **Recommended Questions For Section 3.5:** 18, 19, 20, 25, 28
  Solution of Problem 18 is presented in the following.
  Solution of Problem 20 is explained here: https://bit.ly/2Re7CAm

Solution of Problem 15. (a) Since we have a uniform PDF on the set $\{(x, y)|x^2 + y^2 \le r^2, y \ge 0\}$, it is of the form

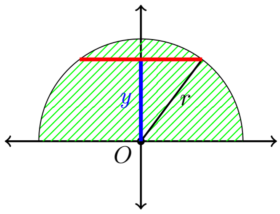

Figure 1.

$$f_{\mathsf{X},\mathsf{Y}}(x, y) = \begin{cases} c & \text{if } x^2 + y^2 \le r^2 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathsf{X},\mathsf{Y}}(x, y)\,\mathrm{d}x\mathrm{d}y = c\tfrac{\pi r^2}{2}$$

Consequently $c = \frac{2}{\pi r^2}$.

(b) If $y \in [0, r]$, then $f_{\mathsf{X},\mathsf{Y}}(x, y) = \frac{2}{\pi r^2}$ for $x \in [-\sqrt{r^2 - y^2}, \sqrt{r^2 - y^2}]$ and $f_{\mathsf{X},\mathsf{Y}}(x, y) = 0$ for $x \notin [-\sqrt{r^2 - y^2}, \sqrt{r^2 - y^2}]$. Furthermore, if either $y<0$ or $y>r$, then $f_{\mathsf{X},\mathsf{Y}}(x, y)=0$ for all x. Thus

$$\begin{aligned} f_{\mathsf{Y}}(y) &= \int_{-\infty}^{\infty} f_{\mathsf{X},\mathsf{Y}}(x, y)\,\mathrm{d}x \\ &= \begin{cases} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2}\mathrm{d}x, & \text{if } y \in [0, r], \\ 0, & \text{if } y \notin [0, r], \end{cases} \\ &= \begin{cases} \frac{4\sqrt{r^2-y^2}}{\pi r^2}, & \text{if } y \in [0, r], \\ 0, & \text{if } y \notin [0, r], \end{cases}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{E}[\mathsf{Y}] &= \int_{-\infty}^{\infty} y f_{\mathsf{Y}}(y)\,\mathrm{d}y \\ &= \int_0^r y\frac{4\sqrt{r^2-y^2}}{\pi r^2}\mathrm{d}y \\ &= \int_0^{\pi/2} r\sin\theta\frac{4r\cos\theta}{\pi r^2} r\cos\theta\mathrm{d}\theta \qquad \text{by change of variables } y = r\sin\theta, \text{ hence } \begin{cases} \mathrm{d}y = r\cos\theta\mathrm{d}\theta \\ \sqrt{r^2 - y^2} = r\cos\theta \end{cases} \\ &= \frac{4r}{\pi}\left(\frac{-\cos^3\theta}{3}\right)\Big|_0^{\pi/2} \\ &= \frac{4r}{3\pi} \end{aligned}$$

(c)

$$\begin{aligned} \mathbf{E}[\mathsf{Y}] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{\mathsf{X},\mathsf{Y}}(x, y)\,\mathrm{d}y\mathrm{d}x \\ &= \int_{-r}^{r}\int_0^{\sqrt{r^2-x^2}} y\frac{2}{\pi r^2}\mathrm{d}y\mathrm{d}x \\ &= \int_{-r}^{r}\left(\frac{y^2}{\pi r^2}\Big|_0^{\sqrt{r^2-x^2}}\right)\mathrm{d}x \\ &= \int_{-r}^{r}\frac{r^2-x^2}{\pi r^2}\mathrm{d}x \\ &= \frac{1}{\pi r^2}\left(r^2x - \frac{x^3}{3}\right)\Big|_{-r}^{r} \\ &= \frac{4r}{3\pi} \end{aligned}$$

□

Solution of Problem 16.

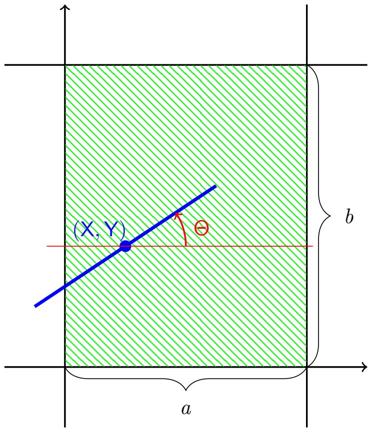

Figure 2.

The needle's midpoint will be in one of the rectangles on the plane. Assuming that the lower-left corner of the rectangle is at the origin and the x-axis of the coordinates is aligned with the horizontal lines, the location of the needle's midpoint in this rectangle can be represented by two random variables X and Y, where X is between 0 and a, and Y is between 0 and b. To describe the position of the needle, we also need to specify the angle Θ that it makes with the horizontal lines, which is between 0 and π radians.
We assume that the random variables $(\mathsf{X}, \mathsf{Y}, \Theta)$ have a uniform distribution on their possible values. Thus then joint PDF of random variables X, Y, and Θ is given by

$$f_{\mathsf{X},\mathsf{Y},\Theta}(x, y, \theta) = \begin{cases} \frac{1}{ab\pi}, & \text{if } x \in [0, a],\ y \in [0, b], \text{ and } \theta \in [0, \pi] \\ 0, & \text{otherwise,} \end{cases},$$

We are asked to calculate the expected number of rectangle sides crossed by the needle and the probability that the needle will cross at least one side of some rectangle. To calculate these quantities will define random variables whose value is one when the needle crosses a specific line and zero otherwise and determine their PMF's of these random variables.

Let Z_γ be a random variable that is equal to 1 if the needle crosses the line $x = \gamma$ and 0 otherwise, i.e.

$$\mathsf{Z}_\gamma := \begin{cases} 1, & \text{if needle crosses } x = \gamma \text{ line,} \\ 0, & \text{otherwise,} \end{cases}.$$

On the other hand, the needle crosses the line $x = \gamma$ if and only if $|\mathsf{X} - \gamma| \le \frac{\ell}{2}|\cos\Theta|$, i.e.

$$\mathsf{Z}_\gamma = \begin{cases} 1, & \text{if } |\mathsf{X} - \gamma| \le \frac{\ell}{2}|\cos\Theta|, \\ 0, & \text{otherwise,} \end{cases}.$$

Then using the fact that $\ell \le a$, we can calculate $\mathbf{P}(\mathsf{Z}_0 = 1)$ as follows

$$\begin{aligned} \mathbf{P}(\mathsf{Z}_0 = 1) &= \int_0^\pi \int_0^{\frac{\ell}{2}|\cos\theta|} \int_0^b \tfrac{1}{ab\pi} \mathrm{d}y \mathrm{d}x \mathrm{d}\theta \\ &= \int_0^\pi \tfrac{\ell}{2a\pi} |\cos\theta| \mathrm{d}\theta \\ &= 2\int_0^{\pi/2} \tfrac{\ell}{2a\pi} \cos\theta \mathrm{d}\theta & \text{because } \int_{\pi/2}^{\pi} \tfrac{\ell}{2a\pi}(-\cos\theta)\mathrm{d}\theta = \int_0^{\pi/2} \tfrac{\ell}{2a\pi}\cos\theta\mathrm{d}\theta \\ &= 2\tfrac{\ell}{2a\pi} \sin\theta\Big|_0^{\pi/2} \\ &= \tfrac{\ell}{a\pi} \end{aligned}$$

Following a similar calculation one can also show $\mathbf{P}(\mathsf{Z}_a = 1) = \ell/a\pi$.

Let W_γ be a random variable that is equal to 1 if the needle crosses the line $y = \gamma$ and 0 otherwise. Then using the fact that the needle crosses the line $y = \gamma$ if and only if $|\mathsf{Y} - \gamma| \le \frac{\ell}{2}\sin\Theta$ and following an analysis similar to the one for Z_0 one can prove that $\mathbf{P}(\mathsf{W}_0 = 1) = \frac{\ell}{b\pi}$ and $\mathbf{P}(\mathsf{W}_b = 1) = \frac{\ell}{b\pi}$.

Since the length of the needle ℓ is less then a, i.e.,$\ell \le a$, a needle whose midpoint is in the shaded area in Figure 2 cannot cross $x = ka$ line for any integer value of k other than $k = 0$ and $k = 1$. Similarly, $\ell \le b$ implies that a needle whose midpoint is in the shaded area can only cross $y = 0$ line and $y = b$ line. Thus the number of rectangle sides crossed by the needle denoted by N is equal to the sum of Z_0, Z_a, W_0, and W_b, i.e.,

$$\mathsf{N} = \mathsf{Z}_0 + \mathsf{Z}_a + \mathsf{W}_0 + \mathsf{W}_b. \tag{1}$$

Hence the expected number of rectangle sides crossed by the needle is given by

$$\begin{aligned} \mathbf{E}[\mathsf{N}] &= \mathbf{E}[\mathsf{Z}_0] + \mathbf{E}[\mathsf{Z}_a] + \mathbf{E}[\mathsf{W}_0] + \mathbf{E}[\mathsf{W}_b] \\ &= \tfrac{2\ell(a+b)}{ab\pi}. \end{aligned}$$

Remark 1. Note that the intersection of the needle with the $x = 0$ line need not to be a side of the rectangle around the shaded area in Figure 2. The intersection might be with a side of a neighboring rectangle, as well. The same is true for the intersection of the needle with the $x = a$ line, the $y = 0$ line, and the $y = b$ line.

Let the random variable C be 1 if the needle crosses at least one side of some rectangle and 0 otherwise. Then for needles whose midpoint are in the shaded in Figure 2, we have the following identity

$$\begin{aligned}\mathsf{C} &= \begin{cases} 0 & \text{if } \mathsf{Z}_0 = \mathsf{Z}_a = \mathsf{W}_0 = \mathsf{W}_b = 0 \\ 1 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 0 & \text{if } \ \mathsf{X} \in \left(\frac{\ell}{2}|\cos\Theta|, a - \frac{\ell}{2}|\cos\Theta|\right) \text{ and } \mathsf{Y} \in \left(\frac{\ell}{2}\sin\Theta, b - \frac{\ell}{2}\sin\Theta\right) \\ 1 & \text{otherwise,} \end{cases}\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{P}(\mathsf{C}=0) &= \int_0^{\pi}\int_{\frac{\ell}{2}|\cos\theta|}^{a-\frac{\ell}{2}|\cos\theta|}\int_{\frac{\ell}{2}\sin\theta}^{b-\frac{\ell}{2}\sin\theta} \tfrac{1}{ab\pi}\mathrm{d}y\mathrm{d}x\mathrm{d}\theta \\ &= \int_0^{\pi} \tfrac{(a-\ell|\cos\theta|)(b-\ell\sin\theta)}{ab\pi}\mathrm{d}\theta \\ &= 2\int_0^{\pi/2} \tfrac{ab-b\ell\cos\theta-a\ell\sin\theta+\ell^2\cos\theta\sin\theta}{ab\pi}\mathrm{d}\theta \qquad \text{because } \int_0^{\pi/2}\tfrac{(a-\ell\cos\theta)(b-\ell\sin\theta)}{ab\pi}\mathrm{d}\theta = \int_{\pi/2}^{\pi}\tfrac{(a+\ell\cos\theta)(b-\ell\sin\theta)}{ab\pi}\mathrm{d}\theta \\ &= 2\left.\tfrac{ab\theta-b\ell\sin\theta+a\ell\cos\theta+\ell^2(1/2)\sin^2\theta}{ab\pi}\right|_0^{\pi/2} \\ &= 1-\tfrac{2b\ell+2a\ell-\ell^2}{ab\pi}\end{aligned}$$

Thus

$$\mathbf{P}(\mathsf{C}=1) = \tfrac{2b\ell+2a\ell-\ell^2}{ab\pi}$$

Remark 2. One obtains the same $\mathbf{E}[\mathsf{N}]$ and $\mathbf{P}(\mathsf{C}=1)$ values by studying the following model:

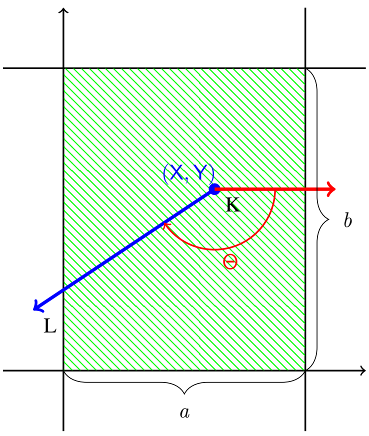

Figure 3.

- The needle has a starting point K and an end point L
- The random variables X and Y are the coordinates of K
- The random variables Θ is the angle between $(1,0)$ vector and the needle.
- The random variables $(\mathsf{X},\mathsf{Y},\Theta)$ has a uniform distribution on their possible values, i.e. the joint PDF of random variables X, Y, and Θ is given by

$$f_{\mathsf{X},\mathsf{Y},\Theta}(x,y,\theta) = \begin{cases} \frac{1}{2\pi ab}, & \text{if } x\in[0,a],\ y\in[0,b], \text{ and } \theta\in[-\pi,\pi]\ , \\ 0, & \text{otherwise,} \end{cases} .$$

□

Solution of Problem 18.

$$f_{\mathsf{X}}(x) = \begin{cases} \frac{x}{4}, & \text{if } 1 < x \leq 3, \\ 0, & \text{otherwise,} \end{cases}, \qquad A = \{\omega | \mathsf{X}(\omega) \geq 2\}.$$

(a) Let us start with calculating the expected value of the continuous random variable X

$$\begin{aligned}\mathbf{E}[\mathsf{X}] &= \int_{-\infty}^{\infty} x f_{\mathsf{X}}(x)\,\mathrm{d}x \\ &= \int_{1}^{3} x \tfrac{x}{4}\mathrm{d}x \\ &= \tfrac{x^3}{12}\Big|_1^3 \\ &= \tfrac{13}{6}\end{aligned}$$

Let us calculate the probability of the event A, i.e., the probability of the event that $\mathsf{X} \geq 2$:

$$\begin{aligned}\mathbf{P}(A) &= \mathbf{P}(\mathsf{X} \geq 2) \\ &= \int_{2}^{\infty} f_{\mathsf{X}}(x)\,\mathrm{d}x \quad \text{because } \mathsf{X} \text{ is a continuous r.v.} \\ &= \int_{2}^{3} \tfrac{x}{4}\mathrm{d}x \\ &= \tfrac{x^2}{8}\Big|_2^3 \\ &= \tfrac{5}{8}\end{aligned}$$

In order to calculate $f_{\mathsf{X|A}}(x)$, we invoke an identity from the textbook, see page 165: For any $\mathrm{B} \subset \mathbb{R}$

$$f_{\mathsf{X}|\{\mathsf{X}\in\mathrm{B}\}}(x) = \begin{cases} \frac{f_{\mathsf{X}}(x)}{\mathbf{P}(\mathsf{X}\in\mathrm{B})} & \text{if } x \in \mathrm{B} \\ 0 & \text{if } x \in \mathrm{B} \end{cases} \tag{2}$$

Note that the event A is not subset of real line; the event A is a subset of the sample space. However, we can express the even A in terms of a subset of the real line, i.e. $[2, \infty)$. Thus using (2) we get

$$\begin{aligned}f_{\mathsf{X|A}}(x) &= f_{\mathsf{X}|\{\mathsf{X}\geq 2\}}(x) \\ &= \begin{cases} \frac{f_{\mathsf{X}}(x)}{\mathbf{P}(\mathsf{X}\geq 2)} & \text{if } x \geq 2 \\ 0 & \text{if } x < 2 \end{cases} \\ &= \begin{cases} \frac{2}{5}x & \text{if } x \in [2,3] \\ 0 & \text{if } x \notin [2,3] \end{cases}\end{aligned}$$

Using the conditional CDF we can calculate the conditional expected value of X

$$\begin{aligned}\mathbf{E}[\mathsf{X}|\,\mathsf{A}] &= \int_{-\infty}^{\infty} f_{\mathsf{X|A}}(x)\,\mathrm{d}x \\ &= \begin{cases} \frac{f_{\mathsf{X}}(x)}{\mathbf{P}(\mathsf{X}\geq 2)} & \text{if } x \geq 2 \\ 0 & \text{if } x < 2 \end{cases} \\ &= \begin{cases} \frac{2}{5}x & \text{if } x \in [2,3] \\ 0 & \text{if } x \notin [2,3] \end{cases}\end{aligned}$$

(b) $\mathsf{Y} = \mathsf{X}^2$. In order to calculate the expected value and the variance of Y we will calculate f_{Y}, i.e., the PDF of Y first. We will learn a commonly used more direct method to calculate f_{Y} using f_{X} in the following chapter. In the following, we rely on the relation between F_{Y} and F_{X} to obtain f_{Y} from f_{X}, instead. Note that

- $\mathbf{P}(\mathsf{Y} < 0) = 0$ because $\mathsf{Y} = \mathsf{X}^2$ and hence $\mathsf{Y} \geq 0$.
- $\mathsf{Y} \in [0, \tau]$ if and only if $\mathsf{X} \in [-\sqrt{\tau}, \sqrt{\tau}]$.

Thus

$$F_{\mathsf{Y}}(\tau) = \begin{cases} 0 & \text{if } \tau < 0 \\ F_{\mathsf{X}}(\sqrt{\tau}) - F_{\mathsf{X}}(-\sqrt{\tau}) + \mathbf{P}(X = -\sqrt{\tau}) & \text{if } \tau \geq 0 \end{cases}.$$

On the other hand,

$$\begin{aligned}F_{\mathsf{X}}(z) &= \int_{-\infty}^{\tau} f_{\mathsf{X}}(x)\,\mathrm{d}x \\ &= \begin{cases} 0, & \text{if } z < 0, \\ \int_1^z \frac{x}{4}\mathrm{d}x, & \text{if } z \in [1,3], \\ 1, & \text{if } z > 3, \end{cases} \\ &= \begin{cases} 0, & \text{if } z < 0, \\ \frac{z^2-1}{8}, & \text{if } z \in [1,3], \\ 1, & \text{if } z > 3, \end{cases}\end{aligned}$$

Thus $\mathbf{P}(X = z) = 0$ for all z and

$$F_{\mathsf{Y}}(\tau) = \begin{cases} 0 & \text{if } \tau < 1 \\ \frac{\tau-1}{8} & \text{if } \tau \in [1,9] \\ 1 & \text{if } \tau > 9 \end{cases}.$$

Then Y is a continuous random variable with the PDF

$$f_{\mathsf{Y}}(\tau) = \begin{cases} \frac{1}{8}, & \text{if } \tau \in [1,9], \\ 0, & \text{if } \tau \notin [1,9], \end{cases}$$

This is because $F_{\mathsf{Y}}(\tau) = \int_{-\infty}^{\tau} f_{\mathsf{Y}}(y)\,\mathrm{d}y$ holds.
Thus Y is uniformly distributed on $[1, 9]$, consequently $\mathbf{E}[\mathsf{Y}] = \frac{9+1}{2} = 5$ and $var(\mathsf{Y}) = \frac{(9-1)^2}{12} = \frac{16}{3}$. Which can also be confirmed without invoking the expressions for the mean and the variance of uniformly distributed random variables as follows:

$$\begin{aligned}\mathbf{E}[\mathsf{Y}] &= \int_{-\infty}^{\infty} y f_{\mathsf{Y}}(y)\mathrm{d}y \\ &= \int_1^9 y\tfrac{1}{8}\mathrm{d}y \\ &= \tfrac{y^2}{16}\Big|_1^9 \\ &= 5\end{aligned}$$

$$\begin{aligned}\mathbf{E}\left[\mathsf{Y}^2\right] &= \int_{-\infty}^{\infty} y^2 f_{\mathsf{Y}}(y)\mathrm{d}y \\ &= \int_1^9 y^2\tfrac{1}{8}\mathrm{d}y \\ &= \tfrac{y^3}{24}\Big|_1^9 \\ &= \tfrac{91}{3}\end{aligned}$$

Thus $var(\mathsf{Y}) = \mathbf{E}\left[\mathsf{Y}^2\right] - \mathbf{E}[\mathsf{Y}]^2 = \frac{16}{3}$.

□