Examples on Series Solutions

1) Write first 4 terms of the series soln. of DE

$$(1-x)y'' + xy' - y = 0 \quad \text{around } x_0 = 0.$$

Soln: $P(x) = 1-x = 0 \Leftrightarrow x=1$ is a SP. All other points are ordinary.

Thus, $x_0 = 0$ ordinary point and the soln. around $x=0$ is

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$(1-x)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n = 0, \quad \text{By equating } \underline{\underline{x^n}}$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n+1}(n+1)(n)x^n + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + na_n - a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} = n(n+1)a_{n+1} - (n-1)a_n, \quad n \geq 1$$

$$2a_2 - a_0 = 0 \qquad a_{n+2} = \frac{n(n+1)a_{n+1} - (n-1)a_n}{(n+2)(n+1)}, \quad n \geq 1$$

$$a_2 = \frac{a_0}{2}$$

$$n=1 \quad a_3 = \frac{2a_2 - 0}{3.2} = \frac{a_2}{3} = \frac{a_0}{6}$$

$$a_2/3 = \frac{a_0}{6}$$

$$n=2 \quad a_4 = \frac{2.3a_3 - a_2}{4.3} = \frac{a_3}{2} - \frac{a_2}{12} = \frac{a_3}{2} - \frac{a_0}{24} = \frac{a_0}{24}$$

$$\cancel{a_4 = \frac{a_0}{8}} \qquad \boxed{a_4 = \frac{a_0}{24}}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = \overbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3}^{\text{first 4 terms}} + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_0}{6}x^3 + \frac{a_0}{24}x^4 + \cdots$$

$$= a_0\left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 \ldots\right] + a_1 x$$

Exm: Find series soln. around $x=1$ for $y'' + (x-1)^2 y' + (x^2-1)y = 0$

$x=1$, O.P.

$t = x-1$ $\qquad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}$

$\frac{dt}{dx} = 1$ $\qquad \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}$

So $\frac{d^2y}{dt^2} + t^2\frac{dy}{dt} + (\overbrace{(t+1)^2-1}^{t^2+2t})y = 0$, $t=0$ OP. so $\underset{\text{soln.}}{y} = \sum_{n=0}^{\infty} a_n t^n$

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + t^2\sum_{n=1}^{\infty} a_n n t^{n-1} + (t^2+2t)\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)t^n + \sum_{n=2}^{\infty} a_{n-1}(n-1)t^n + \sum_{n=2}^{\infty} a_{n-2}t^n + 2\sum_{n=1}^{\infty} a_{n-1}t^n = 0$$

$$2a_2 + 6a_3 t + 2a_0 t + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + a_{n-2} + 2a_{n-1}\right]t^n = 0$$

(1) $2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

(2) $6a_3 + 2a_0 = 0 \Rightarrow \boxed{a_3 = -\frac{a_0}{3}}$

$$a_{n+2} = -\frac{(n+1)a_{n-1} + a_{n-2}}{(n+2)(n+1)}, \quad n \geq 2$$

$\underline{n=2}$: $a_4 = -\frac{3a_1 + a_0}{4.3}$

$\underline{n=3}$: $a_5 = -\frac{3\overset{0}{a_2} - a_1}{5.4} = -\frac{a_1}{5.4}$

$n=4$ $\quad a_6 = -\frac{5a_3 + \overset{0}{a_2}}{6.5} = -\frac{a_3}{6} = \frac{a_0}{6.3}$

$n=5$ $\quad a_7 = \vdots$

So first few terms of the series solns are:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + \overset{0}{a_2} t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots$$

$$= a_0 + a_1 t + \left(\frac{-a_0}{3}\right)t^3 - \frac{3a_1 + a_0}{4.3}t^4 - \frac{a_1}{5.4}t^5 + \frac{a_0}{6.3}t^6 + \cdots$$

$$= a_0\left[1 - \frac{1}{3}t^3 - \frac{1}{4.3}t^4 + \frac{1}{6.3}t^6 + \cdots\right]$$

$$+ a_1\left[t - \frac{1}{4}t^4 - \frac{1}{5.4}t^5 + \cdots\right]$$, since $t = x-1$

$$y(x) = a_0\left[1 - \frac{1}{3}(x-1)^3 - \frac{1}{4.3}(x-1)^4 + \frac{1}{6.3}(x-1)^6 + \cdots\right]$$

$$+ a_1\left[(x-1) - \frac{1}{4}(x-1)^4 - \frac{1}{5.4}(x-1)^5 + \cdots\right]$$

Exm: $2xy'' + 3y' + xy = 0$, find series soln. around $x=0$.

$P(x) = 2x = 0 \Rightarrow x=0$ is a SP. Check $x=0$ is RSP or not.

$$\left.\begin{aligned} \alpha_0 &= \lim_{x\to 0} x\cdot\frac{3}{2x} = \frac{3}{2} \\ \beta_0 &= \lim_{x\to 0} x^2\frac{x}{2x} = 0 \end{aligned}\right\}$$

So $x=0$ is a RSP. Then indicial eqn.

$f(r) = r(r-1) + \frac{3}{2}r = 0 \Rightarrow r_1 = 0,\ r_2 = -\frac{1}{2}$

distinct real roots and $r_1 - r_2 = \frac{1}{2}$ not integer

Since $x=0$ is a RSP, the form of series soln. is

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \ (a_0 \neq 0),\quad y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1},\quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$2\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} + 3\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r} + \sum_{n=1}^{\infty}a_{n-1}x^{n+r} = 0$$

$$2r(r-1)a_0x^{r-1} + 2(r+1)ra_1x^r + 3ra_0x^{r-1} + 3(r+1)a_1x^r$$
$$+ \sum_{n=1}^{\infty}\underbrace{\left[(n+r+1)2(n+r)a_{n+1} + 3(n+r+1)a_{n+1} + a_{n-1}\right]}_{0}x^{n+r} = 0$$

① $(2r(r-1) + 3r)a_0 = 0 \Rightarrow 2r(r-1) + 3r = 0$ "indicial eqn", $a_0 \neq 0$

$\boxed{r_1 = 0}$ $r_2 = -1$

bigger root

② $[2(r+1)r + 3(r+1)]a_1 = 0$

③ $a_{n+1} = -\dfrac{a_{n-1}}{(n+r+1)(2n+2r+3)}$, $n \geq 1$

$\boxed{\text{Start with big root, } r_1 = 0}$

② $\Rightarrow +3a_1 = 0 \Rightarrow \boxed{a_1 = 0}$

③ $\Rightarrow a_{n+1} = -\dfrac{a_{n-1}}{(n+1)(2n+3)}$, $n \geq 1$

$n=1$ $a_2 = -\dfrac{a_0}{2\cdot 5}$

$n=2$ $a_3 = -\dfrac{a_1}{3\cdot 7} = 0$

$n=3$ $a_4 = -\dfrac{a_2}{4\cdot 9} = +\dfrac{a_0}{2\cdot 4\cdot 5\cdot 9}$

$n=4$ $a_5 = -\dfrac{a_3}{5\cdot 11} = 0$

$n=5$ $a_6 = -\dfrac{a_4}{6\cdot 13} = -\dfrac{a_0}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}$

So
$$\boxed{\begin{aligned} a_{2k+1} &= 0, \quad k \geq 0 \\ a_{2k} &= \frac{(-1)^k a_0}{2\cdot 4\cdots(2k)\ 5\cdot 9\cdots(4k+1)}, \quad k \geq 1 \end{aligned}}$$

So, $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \Big|_{r=0} = \sum_{n=0}^{\infty} a_n x^n =$

$$= a_0 + \sum_{k=0}^{\infty} \underbrace{a_{2k+1}}_{0} x^{2k+1} + \sum_{k=1}^{\infty} a_{2k} x^{2k}$$

$$\boxed{y_1(x) = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^k a_0}{2.4\ldots(2k).5.9\ldots(4k+1)} x^{2k}}$$

Since $r_1 - r_2 = \frac{1}{2}$ is not integer, the 2nd soln. will be in the form

$y_2(x) = \sum_{n=0}^{\infty} a_n x^{n+r_2} = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$. To find $y_2(x)$, subs. $r_2 = -\frac{1}{2}$ in ② and ③.

② $\Rightarrow \left[2\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}\right) + 3\left(-\frac{1}{2}+1\right)\right] a_1 = 0 \Rightarrow \boxed{a_1 = 0}$

③ $\Rightarrow \quad a_{n+1} = -\dfrac{a_{n-1}}{\left(n+\frac{1}{2}\right)(2n+2)} = -\dfrac{a_{n-1}}{(2n+1)(n+1)}, \quad n \geq 1$

$n=1 \Rightarrow a_2 = -\dfrac{a_0}{3.2}$

$n=2 \Rightarrow a_3 = -\dfrac{a_1}{5.3} = 0$

$n=3 \Rightarrow a_4 = -\dfrac{a_2}{7.4} = \dfrac{a_0}{7.3.2.4}$

$n=4 \Rightarrow a_5 = \dfrac{-a_3}{\ldots} = 0$

$n=5 \Rightarrow a_6 = \dfrac{-a_4}{11.6} = \dfrac{-a_0}{2.4.6.3.7.11}$

$\Rightarrow \quad a_{2k+1} = 0, \quad k \geq 0$

$a_{2k} = \dfrac{(-1)^k a_0}{2.4\ldots(2k)\,3.7\ldots(4k-1)}$

$$y_2(x) = x^{-1/2} \sum_{n=0}^{\infty} a_n x^n = x^{-1/2}\left[a_0 + \sum_{k=0}^{\infty} \underbrace{a_{2k+1}}_{0} x^{2k+1} + \sum_{k=1}^{\infty} a_{2k} x^{2k}\right]$$

$$= x^{-1/2}\left[a_0 + \sum_{k=1}^{\infty} \frac{(-1)^k a_0}{2.4\ldots(2k)\,3.7\ldots(4k-1)} x^{2k}\right]$$

$$y_2(x) = a_0 x^{-1/2}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2.4\ldots(2k)\,3.7\ldots(4k-1)} x^{2k}\right]$$