Examples on Mechanical and Electrical Vibrations

Note: Be careful on units. Either work with kg and m or lb and ft.

$g = 9.8\ m/s^2 \approx 32\ ft/s^2$

$1 \text{ inch} = \frac{1}{12} ft$

Exm: A mass weighing 3 lb streches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

Note: In this exm. units are taken lb and ft. So write all units in terms of lb and ft

Soln: $mu'' + \gamma u' + ku = F(t)$ is the general eqn. for a forced vibration.

Here damping constant $\gamma = 0$.

There is no external force, i.e. $F(t) = 0$

The spring constant $k = \frac{mg}{l}$ (mass, gravity, $l \to$ alongation)

$$k = \frac{3\,lb}{3\,in} = \frac{3\,lb}{\frac{1}{4}ft} = 12\ lb/ft$$

Mass m, $\underset{\text{weight}}{w} = mg \Rightarrow$ $3\,lb = m \times 9.8\,m/s^2$

$3\,lb = m \times 32\,ft/s^2$

$m = \frac{3}{32}\ ft/s^2$

The the eqn. of motion is

$$\boxed{\frac{3}{32}u'' + 12u = 0} \Rightarrow \boxed{u'' + 128u = 0}$$

The IC's are

$$\boxed{\begin{aligned} u(0) &= -1in = -\frac{1}{12}\ ft \\ u'(0) &= 2ft/s \end{aligned}}$$

Char. eqn: $r^2+128=0 \Rightarrow r_{1,2}=\mp 8\sqrt{2}\,i$.

Then, the general soln is

$$u(t) = A\cos 8\sqrt{2}t + B\sin 8\sqrt{2}t$$

A, B are found by using initial conditions, i.e.

$$u(0) = -\frac{1}{12} = A$$

$$u'(t) = -8\sqrt{2}A\sin 8\sqrt{2}t + 8\sqrt{2}B\cos 8\sqrt{2}t$$

$$u'(0) = 2 = 8\sqrt{2}B \Rightarrow B = \frac{1}{4\sqrt{2}}$$

Thus,
$$u(t) = -\frac{1}{12}\cos 8\sqrt{2}t + \frac{1}{4\sqrt{2}}\sin 8\sqrt{2}t$$

The amplitude $R = \sqrt{A^2+B^2} = \sqrt{\frac{1}{144}+\frac{1}{32}} = \frac{1}{12}\sqrt{\frac{11}{2}}$

The phase of the motion δ, $\tan\delta = \frac{B}{A} = \frac{\frac{1}{\cancel{4}\sqrt{2}}}{-\frac{1}{\cancel{12}_{3}}} = -\frac{3}{\sqrt{2}}$

$$\Rightarrow \delta = -\arctan\left(\frac{3}{\sqrt{2}}\right)$$

The natural frequency $w_0^2 = \frac{k}{m} = \frac{12}{\frac{3}{32}} = 128 \Rightarrow w_0 = \sqrt{128} = 8\sqrt{2}$ rad/s

Period of motion $T = \frac{2\pi}{w_0} = \frac{2\pi}{\sqrt{128}}\,s = \frac{\pi}{4\sqrt{2}}\,s$

So the motion can be writen as

$$u(t) = R\cos(w_0t - \delta)$$

$$u(t) = \frac{1}{12}\sqrt{\frac{11}{2}}\cos\left(\sqrt{128}\,t + \arctan\frac{3}{\sqrt{2}}\right)$$ as given in book.

OR according to our lecture notes. Since,

$u(t) = R\sin(w_0t+\delta)$ where $R = \sqrt{A^2+B^2} = \frac{1}{12}\sqrt{\frac{11}{2}}$

$A = R\sin\delta$, $B = R\cos\delta$ $\Rightarrow$ $\tan\delta = \frac{A}{B} = -\frac{\sqrt{2}}{3} \Rightarrow \delta = \arctan\left(-\frac{\sqrt{2}}{3}\right)$

$w_0 = 8\sqrt{2}$, $T = \pi/4\sqrt{2}$ as above.

$$u(t) = \frac{1}{12}\sqrt{\frac{11}{2}}\sin\left(8\sqrt{2}t - \arctan\frac{\sqrt{2}}{3}\right)$$

Exm. A mass weighing 16 lb streches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb.s/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in/s, find its position u at any time t. Plot u versus t. Determine when the mass first returns to its equilibrium position. Also find the time τ s.t. $|u(t)| < 0.01$ in for all $t > \tau$.

<u>Soln:</u> $\overset{\text{weigh}}{W} = mg \Rightarrow 16\,lb = m \cdot 32\,ft/s^2 \Rightarrow \overset{\text{mass}}{m} = \frac{16}{32} = \frac{1}{2}\,lb\,s^2/ft$

The spring constant, $k = \frac{w}{\ell} = \frac{16\,lb}{3\,in} = \frac{16\,lb}{\frac{1}{4}ft} = 64\,lb/ft$

The damping coefficient, $\gamma = 2\,lb.s/ft$ given in question.

Hence the eqn. of motion is

$$\frac{1}{2}u'' + 2u' + 64u = 0, \quad \left(\begin{array}{l}\text{since } F(t) = 0 \\ \text{i.e. no external force}\end{array}\right)$$

$$\Rightarrow \boxed{u'' + 4u' + 128u = 0}$$

IC's: $u(0) = 0$ (mass set motion from equilibrium)

$u'(0) = 3\,in/s = \frac{1}{4}\,ft/s$ (velocity at equilibrium) (downward so positive)

Char. eqn. $r^2 + 4r + 128 = 0$

$$r_{1,2} = \frac{-4 \mp \sqrt{16 - 4 \cdot 128}}{2} = -2 \mp 2\sqrt{31}\,i = \lambda \mp \mu i \qquad \boxed{\begin{array}{l}\lambda = -2 \\ \mu = 2\sqrt{31}\end{array}}$$

Thus

$$u_{gen} = Ae^{-2t}\cos 2\sqrt{31}t + Be^{-2t}\sin 2\sqrt{31}t$$

$u(0) = 0 \Rightarrow A = 0$

$u'(0) = \frac{1}{4} \Rightarrow u'(t) = -2Be^{-2t}\sin 2\sqrt{31}t + 2\sqrt{31}Be^{-2t}\cos 2\sqrt{31}t$

$u'(0) = 2\sqrt{31}B = \frac{1}{4} \Rightarrow B = \frac{1}{8\sqrt{31}}$

Thus

$$\boxed{u_{gen} = \frac{1}{8\sqrt{31}}e^{-2t}\sin 2\sqrt{31}t} \quad \left(\begin{array}{c}\text{underdamped} \\ \text{motion}\end{array}\right)$$

4

Thus $R = \sqrt{A^2+B^2} = \frac{1}{8\sqrt{31}}$ $\tan\delta = \frac{A}{B} = 0 \Rightarrow \delta = 0$ ← phase

The natural frequency ω_0: $\omega_0 = \sqrt{\frac{k}{m}} = 8\sqrt{2}$

But the frequency of the damped motion is:

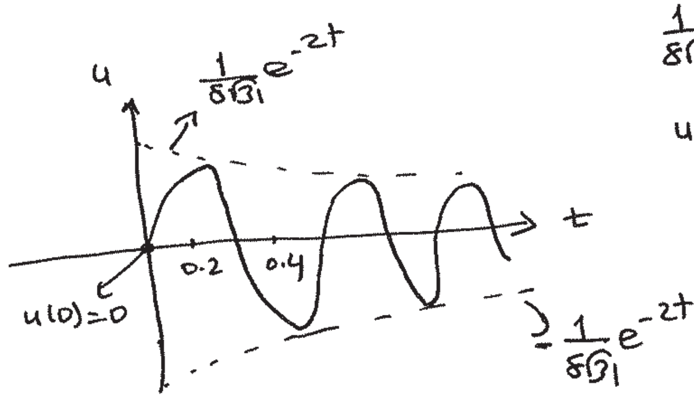
$$\omega = \mu = 2\sqrt{31} \text{ rad/s}$$

Natural period: $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{8\sqrt{2}}$ s.

But the period of damped motion is: $T = \frac{2\pi}{\omega} = \frac{2\pi}{\mu} = \frac{2\pi}{2\sqrt{31}}$ s.

On the other hand:

$u_{gen} \to 0$ as $t \to \infty$. But, it oscillates with period T between the curves $\frac{1}{8\sqrt{31}} e^{-2t}$ and $-\frac{1}{8\sqrt{31}} e^{-2t}$, i.e.



$u(0) = 0$

The mass returns its first equilibrium position, i.e. $t \in [0.2, 0.4]$ as seen from graph: Thus,

$u(t) = 0$, when

$\frac{1}{8\sqrt{31}} \underbrace{e^{-2t}}_{\neq 0} \sin 2\sqrt{31}t = 0 \Rightarrow \sin 2\sqrt{31}t = 0 \Rightarrow 2\sqrt{31}t = \pi \Rightarrow \boxed{t = \frac{\pi}{2\sqrt{31}}}$

On the other hand, $|u(t)| < 0.01$, when

$\left| \frac{1}{8\sqrt{31}} e^{-2t} \sin 2\sqrt{31}t \right| < 0.01$ for $\boxed{t \geq \tau = 0.2145}$

to calculate this we need a calculator.

<u>Note</u> Please study the electric circuits from book by yourself.

Exm: (Forced vibration)

A mass of 5kg streches a spring 10cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2N when the speed of the mass is 4cm/s. If the mass is set in from its equilibrium position with an initial velocity of 3cm/s, formulate the IVP describing the motion of mass.

Soln: (Write all units in N and m)

$m = 5\text{ kg}$

Spring constant, $k = \dfrac{mg}{\ell} = \dfrac{5\text{kg} \times 9.8\text{ m/s}^2}{10\text{ cm}} = \dfrac{5 \times 9.8}{0.1\text{ m}} = 490\text{ N/m}$

Damping coefficient, $\gamma = \dfrac{\overset{\swarrow\text{resistive force}}{F_s}}{\underset{\swarrow\text{velocity}}{u'}} = \dfrac{2\text{N}}{4\text{cm/s}} = \dfrac{2\text{N}}{0.04\text{m/s}} = 50\text{ Ns/m}$

External force, $F(t) = 10\sin\left(\frac{t}{2}\right)$ given.

So eqn. of motion

$$5u'' + 50u' + 490u = 10\sin\left(\tfrac{t}{2}\right) \quad (*)$$

<u>IC's:</u> $u(0) = 0$ (since at start mass is at equilibrium)

$u'(0) = 3\text{cm/s} = 0.03\text{m/s}$ (initial velocity).

$u_{gen} = u_c + u_p$

<u>for u_c:</u> Char. eqn: $5r^2 + 50r + 490 = 0 \Rightarrow r_{1,2} = -5 \mp \sqrt{73}\,i$

$$u_c = Ae^{-5t}\cos\sqrt{73}t + Be^{-5t}\sin\sqrt{73}t$$

<u>for u_p:</u> By undetermined coefficient technique

$u_p = D\cos\frac{t}{2} + E\sin\frac{t}{2}$, D, E: constant.

Find D, E by subs. u_p in eqn. (*).

Thus,

$$u_p = \frac{1}{153281}\left[-160\cos\left(\frac{t}{2}\right) + 3128\sin\left(\frac{t}{2}\right)\right]$$

Hence

$$u_{gen} = u_c(t) + u_p(t)$$

By using IC's, the coefficients A and B in u_c are found as,

$$A = \frac{160}{153281}, \quad B = \frac{383443\sqrt{73}}{1118951300} \quad \&$$

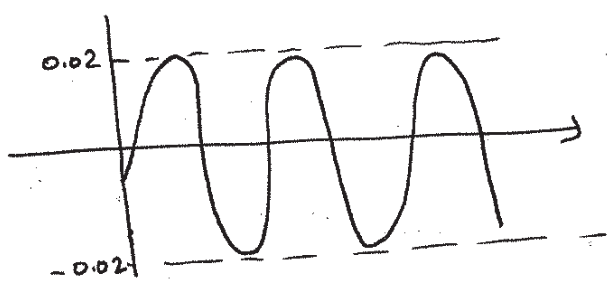
Thus

is transient soln by definition

$$u_{gen} = \frac{1}{153281}\left[160e^{-5t}\cos\sqrt{73}\,t + \frac{383443\sqrt{73}}{7300}e^{-5t}\sin\sqrt{73}\,t\right]$$

$$+ \boxed{u_p(t)} \longrightarrow \text{steady state soln. by definition}$$

The graph of Steady State soln, $u_p(t)$ is



Exm: If an undamped mass-spring system with a mass that weighs 6 lb and a spring constant 1 lb/in is suddenly set in motion at $t=0$ by an external force of $4\cos 7t$ lb determine the position of mass at any time and draw of the displacement versus t.

Soln: The spring constant $k = 1\ \text{lb/in} = 12\ \text{lb/ft}$

$m = \frac{6\ \text{lb}}{32\ \text{ft/s}^2}$, $\gamma = 0$ (undamped), $F(t) = 4\cos 7t$ lb

Then motion is

$$\frac{6}{32}u'' + 12u = 4\cos 7t \Rightarrow \boxed{u'' + 64u = \frac{64}{3}\cos 7t}$$

IC's. $\boxed{\begin{matrix} u(0)=0 \\ u'(0)=0 \end{matrix}}$

$\underline{u_c}$: $r^2 + 64 = 0 \Rightarrow r_{1,2} = \mp 8i$

$u_c = A\cos 8t + B\sin 8t$

$\underline{u_p}$: UC $\Rightarrow$ $u_p = C\cos 7t + D\sin 7t$

$u_p' = -7C\sin 7t + 7D\cos 7t$, $u_p'' = -49C\cos 7t - 49D\sin 7t$

$u_p'' + 64u_p = \frac{64}{3}\cos 7t \Rightarrow -49C\cos 7t - 49D\sin 7t + 64C\cos 7t + 64D\sin 7t = \frac{64}{3}\cos 7t$

$\Rightarrow (64-49)C = \frac{64}{3}$ and $\boxed{D=0}$

$$\boxed{C = \frac{64}{45}}$$

$$\boxed{u_p(t) = \frac{64}{45}\cos 7t}$$

Thus

$u_{gen} = u_c + u_p = A\cos 8t + B\sin 8t + \frac{64}{45}\cos 7t$

with IC's $A = -\frac{64}{45}$, $B=0$

$u_{gen} = -\frac{64}{45}\cos 8t + \frac{64}{45}\cos 7t$

$u_{gen} = \frac{128}{45}\sin\left(\frac{t}{2}\right)\sin\left(\frac{15t}{2}\right)$ by properties of sine and cosine.

Exm: A spring-mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of instantaneous velocity. If the system is driven by an external force of $(3\cos 3t - 2\sin 3t)$ N, determine the steady-state response. Express your answer in the form $R\cos(\omega t - \delta)$

Soln: $k = 3\,N/m$, $m = 2\,kg$, $\gamma = \frac{F_s}{u'} = 1$ (since $F_s = u'$ given)

$F(t) = (3\cos 3t - 2\sin 3t)$ N, thus

$$(*) \quad 2u'' + u' + 3u = 3\cos 3t - 2\sin 3t$$

Since, the system is damped by theory we know that steady state soln. is equal to particular soln.

u_c: $2r^2 + r + 3 = 0 \Rightarrow r_{1,2} = \frac{-1 \mp \sqrt{1-24}}{4} = -\frac{1}{4} \mp \frac{\sqrt{23}}{2} i$

$$u_c = A e^{-\frac{1}{4}t} \cos \frac{\sqrt{23}}{2} t + B e^{-\frac{1}{4}t} \sin \frac{\sqrt{23}}{2} t$$

u_p: By UC: $u_p = C\cos 3t + D\sin 3t$, By substituting u_p in $(*)$ we found $C = -\frac{1}{6}$, $D = \frac{1}{6}$

$$u_p(t) = \frac{1}{6}(\sin 3t - \cos 3t)$$

$R = \sqrt{\frac{1}{6^2} + \frac{1}{6^2}} = \frac{\sqrt{2}}{6}$, $\tan\delta = -1 \Rightarrow \delta = \arctan(-1) = \frac{3\pi}{4}$

Thus $u_p(t) = \frac{\sqrt{2}}{6}\cos\left(3t - \frac{3\pi}{4}\right)$ (according to book)

or

$u_p(t) = \frac{\sqrt{2}}{6}\sin\left(3t + \frac{3\pi}{4}\right)$ (according to our lecture notes)

steady state soln.