Examples on Laplace Transform

1.) $\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$

$= \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a}, \quad s>|a|.$

$$\boxed{\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \quad s>|a|}$$

2.) $\mathcal{L}\{\cosh at\} = \frac{1}{2}\mathcal{L}\{e^{at}+e^{-at}\} = \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right) = \frac{s}{s^2-a^2}, \quad s>|a|$

3.) $\mathcal{L}^{-1}\left\{\frac{1-2s}{s^2+4s+5}\right\} = ?$

$$\frac{1-2s}{s^2+4s+5} = \frac{1}{(s+2)^2+1} - 2\frac{s+2-2}{(s+2)^2+1} = \frac{1}{(s+2)^2+1} - 2\frac{s+2}{(s+2)^2+1} + \frac{4}{(s+2)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1-2s}{s^2+4s+5}\right\} = 5.\ e^{-2t}\sin t - 2e^{-2t}\cos t$$

4.) $\mathcal{L}^{-1}\left\{\frac{4}{(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{2.\ 2!}{(s-1)^{2+1}}\right\} = 2.\ t^2 e^t$

5.) Find Laplace of where $f(t) = 2t$.

a.) $f(t-1)\,u_2(t)$

$f(t-1)\,u_2(t) = (2t-2)\,u_2(t) = 2(t-1-1+1)\,u_2(t) = 2(t-2)u_2(t) + 2u_2(t)$

$\mathcal{L}\{2(t-2)u_2(t) + 2u_2(t)\} = 2.\ e^{-2s}\frac{1}{s^2} + 2\frac{e^{-2s}}{s}$

b.) $f(t) = (t-3)\,u_2(t) - (t-2)\,u_3(t)$, then

$f(t) = (t-2-1)\,u_2(t) - (t-2-1+1)\,u_3(t)$

$f(t) = (t-2)\,u_2(t) - u_2(t) - (t-3)\,u_3(t) - u_3(t)$

$$\mathcal{L}\{f(t)\} = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}$$

6.) Find inverse laplace of

a.) $F(s) = \dfrac{2^{n+1} n!}{s^{n+1}} \Rightarrow f(t) = 2^{n+1} t^n$

b.) $F(s) = \dfrac{1}{9s^2 - 12s + 3} = \dfrac{1}{9\left(s^2 - \frac{12}{9}s + \frac{1}{3}\right)} = \dfrac{1}{9} \cdot \dfrac{1}{\left(s - \frac{2}{3}\right)^2 - \frac{4}{9} + \frac{1}{3}}$

$= \dfrac{1}{9} \dfrac{1}{\left(s - \frac{2}{3}\right)^2 - \frac{1}{9}} = \dfrac{1}{9}\left[\dfrac{\cancel{3}/2}{\left(s-\frac{2}{3}\right) - \frac{1}{3}} - \dfrac{3/2}{\left(s - \frac{2}{3}\right) + \frac{1}{3}}\right]$

$= \dfrac{1}{9} \cdot \dfrac{3}{2}\left(\dfrac{1}{(s - 2/3) - 1/3} - \dfrac{1}{(s - 2/3) + \frac{1}{3}}\right]$

$\mathcal{L}^{-1}\{F(s)\} = f(t) = \dfrac{1}{6} 2 \left[\dfrac{e^{2/3 t} \cdot e^{1/3 t} - e^{2/3 t} \cdot e^{-1/3 t}}{2}\right]$

$= \dfrac{1}{3} e^{2/3 t} \left(\dfrac{e^{1/3 t} - e^{-1/3 t}}{2}\right)$

$$\boxed{f(t) = \frac{1}{3} e^{2/3 t} \sinh\left(\frac{1}{3} t\right)}$$