System of Algebraic Equations

A set of linear algebraic eqns. in n variables

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

can be written as

$$A_{m\times n}\,\vec{x}_{n\times 1} = \vec{b}_{m\times 1}$$

A: coefficient matrix; $\vec{b}$: given

__Defn:__ If $\vec{b}=\vec{0} \Rightarrow$ system is called homogeneous, otherwise it is non homogeneous.

__For square systems:__ $A_{n\times n}$

$Ax=b$

- A nonsingular, $\det A \neq 0$: Unique soln.
  $$Ax=b$$
  $$\underbrace{A^{-1}A}_{I}x = A^{-1}b$$
  $$\boxed{x=A^{-1}b}$$
- A singular, $\det A = 0$: either has no soln. or has soln. but __NOT__ unique.

$Ax=0$

- A nonsingular, $\det A \neq 0$: has only the trivial (zero) soln., i.e. $\vec{x}=\vec{0}$
- A singular, $\det A = 0$: has non-trivial (non zero) solns. beside $\vec{x}=0$ soln.

The soln. of the system $Ax=b$ can be found by Gaussian elimination writing the augmented matrix

$$[A\,|\,b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & b_2 \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right] \xrightarrow[\text{row operations}]{\text{apply}} \left[\begin{array}{c|c} \begin{array}{c}\text{upper}\\ \text{triangular}\\ \text{matrix}\end{array} & \end{array}\right]$$

Defn: A matrix is called to be in row echelon form if it satisfies the following properties

1.) The first nonzero entry of each row is 1 (such elements are called leading 1's)

2.) All entries which are below a leading 1 are 0.

3.) If $i < j$ then the leading 1 on row i must be to the left of the leading 1 on row j.

4.) If there are k rows of 0's in the matrix, they must be the last k rows.

Exm: $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -5 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ are row-reduced echelon

$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is not row echelon

$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ not echelon

Thm: Any matrix can be reduced until a matrix in row echelon form is obtained.

Defn: Number of leading entries is called the rank of matrix A.

Gaussian Elimination

- Form the augmented matrix $[A|b]$
- Apply elementary row operations to $[A|b]$ until A is transformed to a matrix in row echelon form
- The columns that do NOT contain a leading 1 correspond to free variables
- The columns containing leading 1's can be expressed in terms of the free variables.

Exm:
$$x_1 - x_3 + x_4 = 0$$
$$3x_1 + x_2 + x_3 = 1$$
$$-x_1 + x_2 + 2x_3 - x_4 = 2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ -1 & 1 & 2 & -1 & 2 \end{array}\right] \xrightarrow[R_1+R_3]{-3R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & 1 & 1 & 0 & 2 \end{array}\right] \xrightarrow{-R_2+R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & 0 & -3 & 3 & 1 \end{array}\right] \xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & 0 & 1 & -1 & -1/3 \end{array}\right]$$

leading 1's

x_4 : is free

$$\Rightarrow \quad x_1 - x_3 + x_4 = 0$$
$$x_2 + 4x_3 - 3x_4 = 1 \Rightarrow x_2 = \frac{4}{3} - 4x_4 + 3x_4 = \frac{4}{3} - x_4$$
$$x_3 - x_4 = -1/3 \Rightarrow x_3 = -\frac{1}{3} + x_4$$
$$x_1 = x_3 - x_4 = -\frac{1}{3} + x_4 - x_4 = -\frac{1}{3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 4/3 - x_4 \\ -\frac{1}{3} + x_4 \\ x_4 \end{bmatrix}$$

$x_4 \in \mathbb{R}$

Exm:
$$x_1 - x_2 + 3x_3 + x_4 + x_5 = -1$$
$$x_1 - 2x_3 + x_4 + 2x_5 = 3$$
$$2x_1 - 2x_2 + 6x_3 + 4x_5 = 2$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 3 & 1 & 1 & -1 \\ 1 & 0 & -2 & 1 & 2 & 3 \\ 2 & -2 & 6 & 0 & 4 & 2 \end{array}\right] \xrightarrow[-2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 3 & 1 & 1 & -1 \\ 0 & 1 & -5 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 & 2 & 4 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 3 & 1 & 1 & -1 \\ 0 & 1 & -5 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{array}\right]$$

x_1, x_2 and x_4 are leading ones, so x_3 and x_5 are free variables.

$$x_1 - x_2 + 3x_3 + x_4 + x_5 = -1 \Rightarrow x_1 = 5 + 2x_3 - 3x_5$$
$$x_2 - 5x_3 + x_5 = 4 \Rightarrow x_2 = 4 + 5x_3 - x_5$$
$$x_4 - x_5 = -2 \Rightarrow x_4 = -2 + x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 + 2x_3 - 3x_5 \\ 4 + 5x_3 - x_5 \\ x_3 \\ 2 - x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$x_3, x_5 \in \mathbb{R}$.

<u>Linear Independence</u>: A set of n vectors $x^{(1)}, x^{(2)}, \dots, x^{(n)}$ is said to be <u>linearly independent</u> if

$$c_1 x^{(1)} + c x^{(2)} + \dots + c_n x^{(n)} = \vec{0}$$

implies that $c_1 = c_2 = c_3 = \dots = c_n = 0$, otherwise, it is called linearly dependent.

(*) In other words, a set of vectors is linearly independent iff none of the vectors can be expressed as a linear combination of remaining vectors.

$$c_1 \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix} + c_2 \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{bmatrix} + \dots + c_n \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \\ \vdots \\ x_n^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(n)} \\ x_2^{(1)} & \vdots & & \vdots \\ \vdots & & & \vdots \\ x_n^{(1)} & x_n^{(2)} & & x_n^{(n)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = X\vec{c} = \vec{0}$$

"hom. linear system"

So if $\det X \neq 0 \Rightarrow$ only soln. is trivial soln. i.e. $c = 0$

if $\det X = 0 \Rightarrow$ there are nonzero solns.

Thus, $x^{(1)}, \dots, x^{(n)}$ are linearly independent iff $\det X \neq 0$.

Exm: Determine the vectors $x=\begin{pmatrix}1\\2\\-1\end{pmatrix}$, $y=\begin{pmatrix}2\\1\\3\end{pmatrix}$, $z=\begin{bmatrix}-4\\1\\-11\end{bmatrix}$ are linearly independent or not.

$$\begin{bmatrix}1 & 2 & -4\\ 2 & 1 & 1\\ -1 & 3 & -11\end{bmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \det X=-14+42-4\cdot 7=0$$

$\Rightarrow$ linearly dependent.

$$\begin{pmatrix}1 & 2 & -4\\ 2 & 1 & 1\\ -1 & 3 & -11\end{pmatrix}\xrightarrow[R_1+R_3]{-2R_1+R_2}\begin{bmatrix}1 & 2 & -4\\ 0 & -3 & 9\\ 0 & 5 & -15\end{bmatrix}\xrightarrow[\frac{1}{5}R_3]{-\frac{1}{3}R_2}$$

$$\begin{pmatrix}1 & 2 & -4\\ 0 & 1 & -3\\ 0 & 1 & -3\end{pmatrix}\xrightarrow{-R_2+R_3}\begin{pmatrix}1 & 2 & -4\\ 0 & 1 & -3\\ 0 & 0 & 0\end{pmatrix}$$

c_3: free

$c_2=3c_3$

$c_1=-2c_2+4c_3$

$c_1=-2c_3$

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}-2c_3\\3c_3\\c_3\end{pmatrix}=c_3\begin{pmatrix}-2\\3\\1\end{pmatrix}$$ is the non-trivial soln.

Defn: We say that a set of functions $\{f_1(t), f_2(t), \dots, f_n(t)\}$ is linearly independent if the eqn.

$$c_1f_1(t)+c_2f_2(t)+\dots+c_nf_n(t)=0$$

for all t implies $c_1=c_2=\dots=c_n=0$.

Exm: Show that $\{e^t, e^{-t}\}$ are linearly ind.

$$c_1e^t+c_2e^{-t}=0$$ for all t. Choose some specific values of t in order to get a linear system

$t=0 \Rightarrow c_1+c_2=0$

$t=1 \Rightarrow c_1 e + c_2 \frac{1}{e} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ e & \frac{1}{e} \end{bmatrix} \xrightarrow{-eR_1+R_2} \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{e}-e \end{bmatrix} \Rightarrow \left(\frac{1}{e}-e\right)c_2 = 0 \Rightarrow c_2 = 0$$

no free variables

and $c_1+c_2=0 \Rightarrow c_1=0$.

Thus, e^t and e^{-t} are linearly ind.

<u>Remark</u>: There is another way to test a set of funcs. for independence if the funcs. are diff. enough times.

Suppose $f_1, f_2, \dots, f_n$ are funcs. of t which are diff. $n-1$ times.

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0 \quad \text{holds for all } t$$

Differentiate this eqn. repeatedly in order to get a linear system:

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$
$$c_1 f_1' + c_2 f_2' + \dots + c_n f_n' = 0$$
$$\vdots$$
$$c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \dots + c_n f_n^{(n-1)} = 0$$

$$\Rightarrow \overbrace{\begin{bmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{n-1} \end{bmatrix}}^{F} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $\det F \neq 0$ i.e. F is invertible $\Rightarrow c_1 = c_2 = \dots = c_n = 0$

Thus, $f_1, \dots, f_n$ are linearly independent.

Defn:

$$\det F = W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & \cdots & & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of $f_1, f_2, \dots, f_n$.

Thm: If the Wronskian of a set of fncs. is not zero even for one value of t, then the set is linearly independent.

Exm: $\{e^{r_1 t}, e^{r_2 t}\}$ is linearly independent if $r_1 \neq r_2$.

$$W\{e^{r_1 t}, e^{r_2 t}\} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix}$$

$$= e^{(r_1 + r_2) t} (r_2 - r_1) \neq 0 \ \forall t$$

if $r_2 \neq r_1$

$> 0 \ \forall t$

Exercise: show that $\{e^{r_1 t}, \dots, e^{r_n t}\}$ is linearly independent if $r_1, r_2, \dots, r_n$ are distinct.

Eigenvalues and Eigenvectors

The linear system

$$\boxed{A\vec{v} = \lambda\vec{v}} \quad (1)$$

where both $\vec{v} \neq 0$ and scalar λ are unknown, is called an eigenvalue problem. λ is called eigenvalue of A and $\vec{v} \neq \vec{0}$ is a corresponding eigenvector.

Rewrite the system as;

$$(2) \quad \boxed{(A - \lambda I)\vec{v} = 0}$$ "hom. system of eqn."

For a nonzero $\vec{v}$, we must have
$\det(A - \lambda I) = 0$, otherwise $\vec{v} = 0$.

Defn: $\det(A - \lambda I) = 0$ is called the characteristic eqn for A, which is a polynomial in λ. Thus, eigenvalues will be the roots of this characteristic eqn. In general,

$$\det(A - \lambda I) = \text{polynomial of order } n$$
$$= \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0$$

which has n zeros (roots) which are eigenvalues of A.

- Now, let λ be an eigenvalue and then the corresponding eigenvector $\vec{v}$ will be determined from relation (1).

<u>Remark</u> : Clearly if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - \underbrace{(a+d)}_{trace(A)}\lambda + \underbrace{ad-bc}_{\det A} = 0$$

<u>Exm</u>: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$\lambda_1 = -3$, $\lambda_2 = -1$ are the eigenvalues.

<u>Then, find eigenvectors:</u>

<u>$\lambda_1 = -3$</u> $\quad (A-\lambda I)\vec{v} = 0$

$$\begin{pmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{put } \lambda = -3$$

free

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ choose $v_2 = 1$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

<u>$\lambda_2 = -1$</u> $\quad (A-\lambda I)\vec{u} = 0$

$$\begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0} \qquad \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & -2 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

u_1 is free and $u_2 = 0$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $u_1 = 1 \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Note that

$$\underbrace{\begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vec{v} & \vec{u} \\ -1 & 1 \\ 1 & 0 \end{bmatrix}}_{P} = \begin{bmatrix} 3 & -1 \\ -3 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}}_{J}$$

λ_1, λ_2

"Jordan Form"

$$AP = PJ \Rightarrow J = P^{-1}AP$$

That, A is diagonalizable matrix.

<u>Defn</u>: A matrix A is called diagonalizable if $\exists$ an invertible matrix P s.t.

$$J = P^{-1}AP \text{ is a diagonal}$$

matrix. Then

$$A^k = (PJP^{-1})(PJP^{-1})\cdots(PJP^{-1})$$

$$= PJ\underbrace{P^{-1}P}_{I}J\underbrace{P^{-1}P}_{I}JP^{-1}\cdots PJ\underbrace{P^{-1}P}_{I}JP^{-1}$$

$$A^k = PJ^kP^{-1}$$

Exm: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-2) = 0$$

$\underset{\text{multiplicity } 1}{\lambda_1 = 2}$, $\underset{\text{multiplicity } 2}{\lambda_2 = \lambda_3 = -1}$

$\underline{\lambda = 2}$;

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1+R_3]{\frac{1}{2}R_1+R_2} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{bmatrix} \xrightarrow[-\frac{2}{3}R_2]{-\frac{1}{2}R_1,\ R_2+R_3} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x_3$ is free $\Rightarrow$ $x_2 = +x_3$, $x_1 = \frac{1}{2}x_2 + \frac{1}{2}x_3 = x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = -1}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2, x_3 free

$x_1 = -x_2 - x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Note that

$$\underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{P} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix} = \underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{J}$$

$\Rightarrow$ $J = P^{-1}AP$ is a diagonal matrix

"Jordan form"

Exm: Diagonalize the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = \lambda(\lambda-3)^2 = 0$$

$\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 3$

$\underline{\lambda_1 = 0}$: For eigenvector $\vec{u}$ corresponding to $\lambda_1 = 0$

$(A-\lambda_1 I)\vec{u} = 0$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1+R_3]{\frac{1}{2}R_1+R_2} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \xrightarrow[2/3 R_2]{R_2+R_3} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ u_3 is free

$2u_1 - u_2 - u_3 = 0 \Rightarrow 2u_1 = u_2 + u_3 \Rightarrow u_1 = u_3$

$u_2 - u_3 = 0 \Rightarrow u_2 = u_3$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ let $u_3 = 1 \Rightarrow \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\underline{\lambda_2 = \lambda_3 = 3}$:

$(A - \lambda_2 I)\vec{v} = 0$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow[\substack{-R_1+R_3 \\ -R_1}]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ v_2, v_3 are free

$v_1 + v_2 + v_3 = 0 \Rightarrow v_1 = -v_2 - v_3$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_2 - v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

WLOG, let $v_2 = 1$, $v_3 = 0$

$\Rightarrow \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector

Let $v_2 = 0$, $v_3 = 1$ $\Rightarrow \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is another eigenvector

$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent

Note that:

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{P} = \begin{bmatrix} 0 & -3 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{J}$$

$A = PJP^{-1}$ since P is invertible

So A is diagonalizable

Exm: $A = \begin{pmatrix} 5 & 0 & 4 \\ -6 & 1 & -1 \\ -6 & 0 & -5 \end{pmatrix}$

$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 0 & 4 \\ -6 & 1-\lambda & -1 \\ -6 & 0 & -5-\lambda \end{vmatrix} = (5-\lambda)\left[(1-\lambda)(-5-\lambda)\right] + 4\left[0+6(1-\lambda)\right] = 0$

$(5-\lambda)(1-\lambda)(-5-\lambda) + 24(1-\lambda) = 0$

$(1-\lambda)\left(-25+\lambda^2+24\right) = 0 \Rightarrow (1-\lambda)(\lambda^2-1) = 0$

$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = -1$

For eigenvectors:

$\lambda_1 = \lambda_2 = 1$: $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$(A-\lambda_1 I)\vec{v} = 0$

$\begin{pmatrix} 4 & 0 & 4 \\ -6 & 0 & -1 \\ -6 & 0 & -6 \end{pmatrix} \xrightarrow[\frac{1}{6}R_2,\ \frac{1}{6}R_3]{\frac{1}{4}R_1} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1/6 \\ -1 & 0 & -1 \end{bmatrix} \xrightarrow[R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 5/6 \\ 0 & 0 & 0 \end{bmatrix}$ v_2: free

$v_1 + v_3 = 0 \Rightarrow v_1 = -v_3 = 0 \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\frac{5}{6} v_3 = 0 \Rightarrow v_3 = 0$

let $v_2 = 1 \Rightarrow \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ only one eigenvector

$\lambda_3 = -1$ $(A-\lambda_3 I)\vec{u} = 0$ $\vec{u}_3$: free (one nonzero soln)

$\begin{pmatrix} 6 & 0 & 4 \\ -6 & 2 & -1 \\ -6 & 0 & -4 \end{pmatrix} \xrightarrow[R_1+R_3]{R_1+R_2} \begin{pmatrix} 6 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$6u_1 + 4u_3 = 0$

$2u_2 + 3u_3 = 0$

$u_1 = -\frac{2}{3}u_3$
$u_2 = -\frac{3}{2}u_3$ $\Rightarrow \vec{u} = \begin{pmatrix} -\frac{2}{3}u_3 \\ -\frac{3}{2}u_3 \\ u_3 \end{pmatrix} = u_3 \begin{pmatrix} -2/3 \\ -3/2 \\ 1 \end{pmatrix}$

let $u_3 = 1 \Rightarrow \vec{u} = \begin{pmatrix} -2/3 \\ -3/2 \\ 1 \end{pmatrix}$

There are only 2 lin. ind. eigenvectors.

$\therefore$ A is not diagonalizable