Question: Is every matrix diagonalizable? No!
e.g. complex eigenvalues or repeated roots with multiplicity ≥ 3!

Question: What if A is NOT diagonalizable?

Jordan Forms:

Thm: For every matrix A, there exists an invertible matrix P (possibly complex valued) s.t.

$$\underline{P} A P = J$$

where J is in Jordan form.

Jordan Form

$$\left[\begin{array}{cccc|cccc|c|ccc}
\lambda_1 & 1 & & 0 & & & & & & & & \\
 & \lambda_1 & 1 & & & & & & & & & \\
0 & & \ddots & 1 & & & & & & & & \\
 & & & \lambda_1 & & & & & & & & \\
\hline
 & & & & \lambda_2 & 1 & & 0 & & & & \\
 & & & & & \ddots & \ddots & & & & & \\
 & & & & 0 & & & 1 & & & & \\
 & & & & & & & \lambda_2 & & & & \\
\hline
 & & & & & & & & \ddots & & & \\
\hline
 & & & & & & & & & \lambda_n & 1 & 0 \\
 & & & & & & & & & 0 & \ddots & 1 \\
 & & & & & & & & & & & \lambda_n
\end{array}\right]$$

Jordan blocks

Exm: $A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$ $\quad \lambda = 1, 2, 4, 4$

$$J = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 4 & 1 \\ & & & 4 \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 4 & \\ & & & 4 \end{bmatrix}$$

Question: How do we find P in such a case?

Let us concentrate on one Jordan block.

$$AP = PJ$$

$$A[\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n] = (\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n) \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

$$= [\lambda \vec{v}_1 | \vec{v}_1 + \lambda \vec{v}_2 | \vec{v}_2 + \lambda \vec{v}_3 | \dots | \vec{v}_{n-1} + \lambda \vec{v}_n]$$

$$A\vec{v}_1 = \lambda \vec{v}_1 \longrightarrow (A-\lambda I)\vec{v}_1 = 0$$
$$A\vec{v}_2 = \vec{v}_1 + \lambda \vec{v}_2 \longrightarrow (A-\lambda I)\vec{v}_2 = \vec{v}_1$$
$$A\vec{v}_3 = \vec{v}_2 + \lambda \vec{v}_3 \longrightarrow (A-\lambda I)\vec{v}_3 = \vec{v}_2$$
$$\vdots \qquad\qquad \vdots$$
$$A\vec{v}_n = \vec{v}_{n-1} + \lambda \vec{v}_n \longrightarrow (A-\lambda I)\vec{v}_n = \vec{v}_{n-1}$$

$\vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$ are called <u>generalized eigenvectors</u> corresponding to λ.

Starting from $\vec{v}_1$ (eigenvector corresponding to λ) find all generalized eigenvectors.

<u>Remark</u>:

# of linearly independent eigenvectors corresponding an eigenvalue λ = # of Jordan blocks in Jordan form corresponding to eigenvalue λ

multiplicity of eigenvalue λ = size of Jordan form corresponding to λ

Exm: $\lambda(\lambda-1)^2(\lambda-2)=0$

$\lambda_1=0$, $\underbrace{\lambda_2=1}_{\text{multiplicity } 2}$, $\lambda_3=2$. The possibilities for Jordan form:

$J_1=[0]$, $J_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ or $J_2=\begin{bmatrix}1&1\\0&1\end{bmatrix}$

$J_3=[2]$

$$J=\begin{bmatrix}J_1&&\\&J_2&\\&&J_3\end{bmatrix}=\begin{bmatrix}0&&&\\&1&0&\\&0&1&\\&&&2\end{bmatrix}$$

or $$J=\begin{bmatrix}J_1&&\\&J_2&\\&&J_3\end{bmatrix}=\begin{bmatrix}0&&&\\&1&1&\\&0&1&\\&&&2\end{bmatrix}$$

Exm: $(\lambda-1)^2(\lambda-2)^3=0$

$\lambda_1=1$ multiplicity 2 (size of Jordan form 2×2)

$J_1=\begin{bmatrix}1&0\\0&1\end{bmatrix}_{2\times2}$ or $J_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}_{2\times2}$

$\downarrow$ 2 eigenvectors = 2 Jordan blocks

$\downarrow$ 1 eigenvector = 1 Jordan block

$$J=\begin{bmatrix}J_1&\\&J_2\end{bmatrix}_{5\times5}$$

$J_2=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$

$\downarrow$ 1 eigenvector = 1 block

$\lambda_2=2$ multiplicity 3 (size of Jordan block 3×3)

$J_2=\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}$ or $J_2=\begin{bmatrix}2&1&\\&2&\\&&2\end{bmatrix}$ or $J_2=\begin{bmatrix}2&&\\&2&1\\&&2\end{bmatrix}$ or

$\downarrow$ 3 eigenvectors = 3 blocks

2 eigenvectors = 2 blocks

Exm: Find the Jordan form of

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}$$

$\det(A-\lambda I) = (2-\lambda)^3 = 0$

multiplicity 3
$\parallel$
size of Jordan form

Eigenvectors: $\vec{v}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

a_3: free
a_1, a_2: leading

$a_1 - a_2 - a_3 = 0 \Rightarrow a_1 = a_2 + a_3 = 0$
$a_2 = -a_3$

$\vec{v} = a_3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\vec{v}^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ only one eigenvector = 1 Jordan block

$$\Rightarrow J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Let show it by finding generalized eigenvectors $\vec{v}^{(2)}$ and $\vec{v}^{(3)}$.

$(A-\lambda I)\vec{v}^{(2)} = \vec{v}^{(1)}$, $\quad \vec{v}^{(2)} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & -1 \\ -3 & 2 & 2 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{2R_1+R_2} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-b_1 + b_2 + b_3 = 0 \Rightarrow b_1 = b_2 + b_3 = -1 - b_3 + b_3 = -1$
$b_2 + b_3 = -1 \Rightarrow b_2 = -1 - b_3$

$\vec{v}^{(2)} = \begin{bmatrix} -1 \\ -1-b_3 \\ b_3 \end{bmatrix}$ Let $b_3 = 0 \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

For $\vec{v}^{(3)} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, let solve $(A-\lambda I)\vec{v}^{(3)} = \vec{v}^{(2)}$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 2 & -1 & -1 & -1 \\ -3 & 2 & 2 & 0 \end{array}\right] \xrightarrow[-3R_1+R_2]{2R_1+R_2} \left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -1 & 3 \end{array}\right] \xrightarrow{R_2+R_3}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-c_1+c_2+c_3=-1 = -3-c_3+c_3+1=-2$$
$$c_2+c_3=-3 \Rightarrow c_2=-3-c_3$$

$\vec{v}^{(3)} = \begin{bmatrix} -2 \\ -3-c_3 \\ c_3 \end{bmatrix}$ choose $c_3=0 \Rightarrow \vec{v}^{(3)} = \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}$

Thus, $P = \begin{bmatrix} \vec{v}^{(1)} & \vec{v}^{(2)} & \vec{v}^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ -1 & -1 & -3 \\ 1 & 0 & 0 \end{bmatrix}$

$$P^{-1}AP = J$$

For P^{-1}:

$$\left[\begin{array}{ccc|ccc} 0 & -1 & -2 & 1 & 0 & 0 \\ -1 & -1 & -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -3 & 0 & 1 & 1 \\ 0 & -1 & -2 & 1 & 0 & 0 \end{array}\right] \xrightarrow[-R_2]{-R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \xrightarrow{-3R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 0 & 0 & 1 \\ -3 & 2 & 2 \\ 1 & -1 & -1 \end{matrix}}_{P^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$J = \begin{bmatrix} 0 & 0 & 1 \\ -3 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ -1 & -1 & -3 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol'n for $\vec{x}' = Ax \Rightarrow \vec{x} = e^{At}\vec{c}$

$A = PJP^{-1}$ $\quad \vec{x} = Pe^{Jt}P^{-1}\vec{c}$ $\Big\}$ what is e^{Jt}?

Exponentiating a matrix in Jordan form

If $A = PJP^{-1}$ where P is invertible matrix;

then $e^{At} = P e^{Jt} P^{-1}$. How do we find e^{Jt}.

Enough to do one Jordan block at a time.

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix} \Rightarrow e^{Jt} = \begin{bmatrix} e^{J_1 t} & & & \\ & e^{J_2 t} & & \\ & & \ddots & \\ & & & e^{J_k t} \end{bmatrix}$$

Thm: $J = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ 0 & & & \lambda \end{bmatrix} \Rightarrow$

$$e^{Jt} = e^{\lambda t} \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ 0 & 1 & t & \cdots & t^{n-2}/(n-2)! \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & t^2/2! \\ & & & \ddots & t \\ & & & & 1 \end{bmatrix}$$

<u>Remark</u> Using this formula we can solve any constant coefficient hom. system.

Solution to $\vec{x}\,' = A\vec{x}$ will be

$$\vec{x} = e^{At}\vec{c} \quad \text{or} \quad \vec{x} = \underbrace{e^{At}P}\,\vec{c}$$

is also a fun. soln. for P is invertible.

$$\vec{x} = P e^{Jt} P^{-1}\vec{c} \quad \text{or} \quad \vec{x} = P e^{Jt} \overset{I}{P^{-1}P}\,\vec{c}$$

$$\vec{x} = P e^{Jt}\vec{c}$$

Exm:
$$J = \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 2 & 1 & 0 \\ & 0 & 2 & 1 \\ & 0 & 0 & 2 \end{array}\right] \qquad J_1 = 1 \qquad J_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$e^{J_1 t} = [e^t]$$

$$e^{J_2 t} = e^{2t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{t} \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & \\ & e^{J_2 t} \end{bmatrix} = \left[\begin{array}{c|ccc} e^t & 0 & 0 & 0 \\ \hline 0 & e^{2t} & te^{2t} & t^2/2\, e^{2t} \\ 0 & 0 & e^{2t} & te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{array}\right]$$

Exm: Thus the soln. for the previous example

$$\vec{x}' = \overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}}^{A} \vec{x} \quad \text{is}$$

$$\vec{x} = \underbrace{P e^{Jt} P^{-1}}_{e^{At}} \vec{c} \quad \text{where} \quad e^{Jt} = e^{2t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -1 & -2 \\ -1 & -1 & -3 \\ 1 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -3 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$e^{At} = P e^{Jt} P^{-1} = \begin{bmatrix} 0 & -1 & -2 \\ -1 & -1 & -3 \\ 1 & 0 & 0 \end{bmatrix} e^{2t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -3 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -t+1 & t & t \\ -\frac{t^2}{2}+2t & \frac{t^2}{2}+3t+1 & \frac{t^2}{2}-t \\ t^2/2-3t & t & 1+2t-t^2/2 \end{bmatrix}$$

Note here that $e^{At}\Big|_{t=0} = I$.

$$\vec{x} = e^{At}\vec{c} = c_1 e^{2t}\begin{bmatrix} -t+1 \\ -\frac{t^2}{2}+2t \\ \frac{t^2}{2}-3t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ t^2/2+3t+1 \\ t \end{bmatrix}$$

$$+ c_3 e^{2t}\begin{bmatrix} t \\ t^2/2 - t \\ 1+2t-t^2/2 \end{bmatrix}$$

Exm: $\vec{x}' = \begin{bmatrix} -2 & 0 \\ 4 & -2 \end{bmatrix} \vec{x}$ Solve !

$$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 0 \\ 4 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 = 0$$

$\lambda_{1,2} = -2$
multiplicity 2
"
size of Jordan form for $\lambda = 2$.

$\underline{\lambda = 2}$: $(A+2I)\vec{v}^{(1)} = 0 \qquad \vec{v}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad a_1 = 0,\ a_2 : \text{free}$$

$$\vec{v}^{(1)} = a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{let } a_2 = 1 \Rightarrow \vec{v}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

one eigenvector = one Jordan block

Thus, the Jordan form must be $J = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

Find P:

For $\vec{v}^{(2)}$: $(A+2I)\vec{v}^{(2)} = \vec{v}^{(1)}$ where $\vec{v}^{(2)} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ generalized eigenvector

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 4 & 0 & 1 \end{array}\right] \Rightarrow \begin{array}{l} 4b_1 = 1 \Rightarrow b_1 = 1/4 \\ b_2 : \text{free} \end{array}$$

$$\vec{v}^{(2)} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ b_2 \end{bmatrix}, \quad \text{Let } b_2 = 0 \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} \vec{v}^{(1)} & \vec{v}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

Thus, $J = P^{-1}AP = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

Then, the general soln:

$$\vec{x}_{gen} = e^{At}\vec{c} = e^{PJP^{-1}t}\vec{c}$$

$$= Pe^{Jt}P^{-1}\vec{c}$$

$$= \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \end{bmatrix} e^{-2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= e^{-2t} \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4t & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= e^{-2t} \begin{bmatrix} 1 & 0 \\ 4t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= c_1 \underbrace{e^{-2t}\begin{bmatrix} 1 \\ 4t \end{bmatrix}}_{\vec{x}_1} + c_2 \underbrace{e^{-2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{x}_2}$$

linearly independent solutions.

Remark: For 2×2 systems : $\lambda = \lambda_1 = \lambda_2$

Then, if $\exists$ only one eigenvector $\vec{v}^{(1)}$, then one solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x}_1(t) = e^{\lambda t}\vec{v}^{(1)}$$

where $\vec{v}^{(1)}$ satisfies

$$(A - \lambda I)\vec{v}^{(1)} = \vec{0}$$

Then, for the second solution $\vec{x}_2$, the form of the solution is

$$\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v}^{(1)} + \vec{v}^{(2)}\right)$$

Substitute in $\vec{x}' = A\vec{x}$

$$\vec{x}_2{}' = \lambda e^{\lambda t}\left(t\vec{v}^{(1)} + \vec{v}^{(2)}\right) + e^{\lambda t}\vec{v}^{(1)}$$

$\vec{x}_2{}' = A\vec{x}_2 \Rightarrow$

$$\lambda e^{\lambda t}\left(t\vec{v}^{(1)} + \vec{v}^{(2)}\right) + e^{\lambda t}\vec{v}^{(1)} = Ae^{\lambda t}\left(t\vec{v}^{(1)} + \vec{v}^{(2)}\right)$$

$$\cancel{\lambda e^{\lambda t} t\vec{v}^{(1)}} + \lambda e^{\lambda t}\vec{v}^{(2)} + e^{\lambda t}\vec{v}^{(1)} = \cancel{\underbrace{Ae^{\lambda t} t\vec{v}^{(1)}}_{e^{\lambda t} t \underbrace{A\vec{v}^{(1)}}_{\lambda \vec{v}^{(1)}}}} + Ae^{\lambda t}\vec{v}^{(2)}$$

$$\lambda\vec{v}^{(2)} + \vec{v}^{(1)} = A\vec{v}^{(2)}$$

$$\boxed{(A - \lambda I)\vec{v}^{(2)} = \vec{v}^{(1)}}$$

Solve for $\vec{v}^{(2)}$ which is generalized eigenvector

Moreover, $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $e^{Jt} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

and $P = [\vec{v}^{(1)} \ \vec{v}^{(2)}]$ and the fun. soln.

$$e^{At} = Pe^{Jt}P^{-1} = [\vec{v}^{(1)} \ \vec{v}^{(2)}] \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} (\vec{v}^{(1)} \ \vec{v}^{(2)})^{-1}$$

On the other hand, $e^{At}P$ is also a fun. matrix

thens

$$\Phi(t) = Pe^{Jt}\underbrace{P^{-1}P}_{I} = Pe^{Jt}.$$

Thus, the solution will be

$$\vec{x} = \Phi(t)\vec{c}$$

$$= Pe^{Jt}\vec{c}.$$

<u>Exm</u>: Solve $\vec{x}' = \begin{bmatrix} 2 & 4 & -8 \\ 0 & 0 & 4 \\ 0 & -1 & 4 \end{bmatrix} \vec{x}$

Soln;

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 4 & -8 \\ 0 & -\lambda & 4 \\ 0 & -1 & 4-\lambda \end{vmatrix} = -(\lambda-2)^3 = 0$$

$\lambda = 2$ multiplicity 3 = size of Jordan form

<u>Eigenvectors</u>: $(A-2I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & 4 & -8 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$v_2 - 2v_3 = 0$

$v_2 = 2v_3$

$\vec{v} = v_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$E = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$.

Choose $\vec{v}^{(3)}$ which satisfies

$$(A-2I)^2 \vec{v}^{(3)} = 0 \quad \text{and} \quad (A-2I)\vec{v}^{(3)} \neq 0$$

Here $(A-2I)^2 = 0$, so $\vec{v}^{(3)}$ can be any vector independent of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$. Thus, choose WLOG $\vec{v}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Then $\vec{v}^{(2)} = (A-2I)\vec{v}^{(3)} = \begin{bmatrix} 0 & 4 & -8 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}$

Finally, $\vec{v}^{(1)}$ is in the set E but independent of $\vec{v}^{(2)} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}$, say WLOG $\vec{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Thus, $P = \begin{bmatrix} v^{(1)} & v^{(2)} & v^{(3)} \end{bmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

<u>For P^{-1}:</u>

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 4 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{matrix}}_{P^{-1}}$$

$$J = P^{-1}AP = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 4 & -8 \\ 0 & 0 & 4 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right)$$

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & \\ & e^{J_2 t} \end{bmatrix} = \left[\begin{array}{c|cc} e^{2t} & 0 & 0 \\ \hline 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{array}\right]$$

$$e^{At} = Pe^{Jt}P^{-1} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{pmatrix} e^{2t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$= e^{2t} \begin{bmatrix} 1 & 4t & -8t \\ 0 & -2t+1 & 4t \\ 0 & -t & 1+2t \end{bmatrix}$$

Note that $e^{At}\big|_{t=0} = I$

$$\vec{x}_{gen} = e^{At}\vec{c} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 4t \\ -2t+1 \\ -t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -8t \\ 4t \\ 1+2t \end{bmatrix}$$

Exercise: Solve $\vec{x}\,' = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix} \vec{x}$

$\det(A - \lambda I) = (\lambda - 1)^3 = 0 \Rightarrow \lambda = 1$ eigenvalue multiplicity 3

Eigenvectors: $(A - I)\vec{v} = 0$

$$E = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Choose $\vec{v}^{(3)}$ s.t. $(A - I)^2 \vec{v}^{(3)} = 0$ but

$(A - I)\vec{v}^{(3)} \neq 0$ (i.e. $\vec{v}^{(3)}$ is not an eigenvector in E)

$(A - I)^2 = 0 \Rightarrow \vec{v}^{(3)}$ can be any vector linearly independent of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

Choose WLOG $\vec{v}^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Then $\vec{v}^{(2)} = (A - I)\vec{v}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Finally, $\vec{v}^{(1)}$ should be any eigenvector which is independent of $\vec{v}^{(2)}$ but in E.
choose WLOG $\vec{v}^{(1)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Then $P = \left[\vec{v}^{(1)}\ \vec{v}^{(2)}\ \vec{v}^{(3)}\right] = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ and $J = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

OR

$$P = \left[\vec{v}^{(2)} \ \vec{v}^{(3)} \ \vec{v}^{(1)}\right] = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

related should be written together

$$\Rightarrow P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow J = P^{-1}AP = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$e^{At} = e^{PJP^{-1}t} = P e^{Jt} P^{-1}$$

$$= \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} e^{t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= e^{t} \begin{pmatrix} 1+t & 2t & 3t \\ t & 1+2t & 3t \\ -t & -2t & 1-3t \end{pmatrix}$$

$$\vec{x}_{gen} = c_1 \underbrace{e^{t} \begin{pmatrix} 1+t \\ t \\ -t \end{pmatrix}}_{= \vec{x}_1(t)} + c_2 \underbrace{e^{t} \begin{pmatrix} 2t \\ 1+2t \\ -2t \end{pmatrix}}_{= \vec{x}_2(t)} + c_3 \underbrace{e^{t} \begin{pmatrix} 3t \\ 3t \\ 1-3t \end{pmatrix}}_{= \vec{x}_3(t)}$$

Exm: $A = \begin{bmatrix} 5 & -1 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 12 & -3 \end{bmatrix}$ Find e^{Jt} where $J = P^{-1}AP$ with invertible P.

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -1 & 0 & 0 \\ 9 & -1-\lambda & 0 & 0 \\ 0 & 0 & 7-\lambda & -2 \\ 0 & 0 & 12 & -3-\lambda \end{vmatrix} = (\lambda-2)^2(\lambda-3)(\lambda-1) = 0$$

$\lambda_1 = 2$ multiplicity 2, $\lambda_2 = 3$, $\lambda_3 = 1$

$\underline{\lambda_1 = 2}$: $(A - 2I)\vec{v} = 0$

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 12 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

v_2: free $\Rightarrow$ 1 linearly ind. eigenvector

$\Downarrow$

one Jordan block for $\lambda_1 = 2$ of size 2

$$\Rightarrow J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

Thus, Jordan form of A:

$$J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\to$ eigenvector: $3v_1 - v_2 = 0 \Rightarrow 3v_1 = v_2$

$5v_3 - 2v_4 = 0 \Rightarrow v_3 = 0$

$-v_4 = 0$

$$\vec{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{v}^{(2)}$ is generalized eigenvector obtained by

$(A-2I)\vec{v}^{(2)} = \vec{v}^{(1)} \Rightarrow$

$$\left[\begin{array}{cccc|c} 3 & -1 & 0 & 0 & 1 \\ 9 & -3 & 0 & 0 & 3 \\ 0 & 0 & 5 & -2 & 0 \\ 0 & 0 & 12 & -5 & 0 \end{array}\right] \xrightarrow{\text{solve}} \vec{v}^{(2)} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda_2 = 3}$: $(A-3I)\vec{v}^{(3)} = 0 \to \vec{v}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}$

$\underline{\lambda_3 = 1}$: $(A-I)\vec{v}^{(4)} = 0 \longrightarrow \vec{v}^{(4)} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix}$

$$P = \left[\vec{v}^{(1)}\ \vec{v}^{(2)}\ \vec{v}^{(3)}\ \vec{v}^{(4)}\right] = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

$$J = P^{-1}AP = \left(\begin{array}{cc|c|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$e^{Jt} = \begin{pmatrix} e^{J_1 t} & & \\ & e^{J_2 t} & \\ & & e^{J_3 t} \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{3t} & 0 \\ 0 & 0 & 0 & e^{t} \end{pmatrix}$$

Since $e^{At}P = Pe^{Jt}P^{-1}P = Pe^{Jt}$ is a fn. matrix,

$$\vec{x}_{gen} = e^{At}P\vec{c} = Pe^{Jt}\vec{c}$$