11. Digraphs

The concept of digraphs (or directed graphs) is one of the richest theories in graph theory, mainly because of their applications to physical problems. For example, flow networks with valves in the pipes and electrical networks are represented by digraphs. They are applied in abstract representations of computer programs and are an invaluable tools in the study of sequential machines. They are also used for systems analysis in control theory. Most of the concepts and terminology of undirected graphs are also applicable to digraphs, and hence in this chapter more emphasis will be given to those properties of digraphs that are not found in undirected graphs.

There are two different treatments of digraphs—one can be found in the book by Harray, Norman and Cartwright [191] and the other in the book by Berge [18]. The former discusses the application of digraphs to sociological problems, and the latter gives a comprehensive mathematical treatment. One can also refer to the books by N. Deo [63], Harary [104], Behzad, Chartrand and Lesniak-Foster [16], and Buckley and Harary [42].

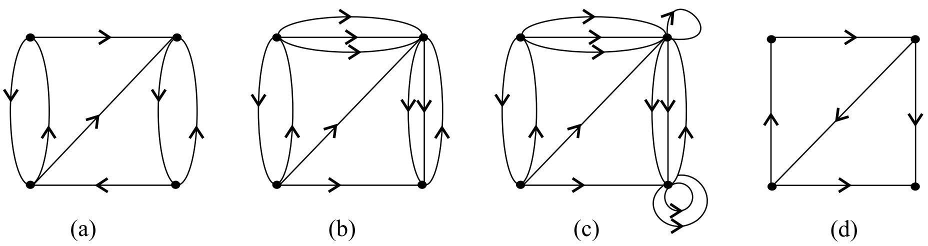
11.1 Basic Definitions

Digraphs (Directed graphs): A digraph D is a pair (V, A), where V is a nonempty set whose elements are called the *vertices* and A is the subset of the set of ordered pairs of distinct elements of V. The elements of A are called the *arcs* of D (Fig. 11.1(a)).

Multidigraphs: A multidigraph D is a pair (V, A), where V is a nonempty set of vertices and A is a multiset of arcs, which is a multisubset of the set of ordered pairs of distinct elements of V. The number of times an arc occurs in D is called its *multiplicity* and arcs with multiplicity greater than one are called *multiple arcs* of D (Fig. 11.1(b)).

General digraphs: A general digraph D is a pair (V, A), where V is a nonempty set of vertices, and A is a multiset of arcs, which is a multisubset of the cartesian product of V with itself. An arc of the form uu is called a *loop* of D and arcs which are not loops are called *proper arcs* of D. The number of times an arc occurs is called its multiplicity. A loop with multiplicity greater than one is called a *multiple loop* (Fig. 11.1(c)).

Oriented graph: A digraph containing no symmetric pair of arcs is called an oriented graph (Fig. 11.1(d)).



(a) (b) (c) (d)

Fig. 11.1

For $u,\ v \in V$, an arc $a = (u,\ v) \in$ A is denoted by uv and implies that a is directed from u to v. Here, u is the *initial* vertex (tail) and v is the *terminal* vertex (head). Also we say that a joins u to v; a is incident with u and v; a is incident from u and a is incident to v; and u is adjacent to v and v is adjacent from u. In case both uv and vu belong to A, then uv and vu are called a *symmetric pair of arcs*. For any $v \in V$, the number of arcs incident to v is the *indegree* of v and is denoted by $d^-(v)$. The number of arcs incident from v is called the *outdegree* of v and is denoted by $d^+(v)$. Berge calls indegree and outdegree as *inner* and *outer-demi degrees*. The *total degree* (or simply degree) of v is $d(v) = d^-(v) + d^+(v)$.

We define $N^+(v)$ and $N^-(v)$ by

$$N^+(v) = \{u \in V : vu \in A\} \text{ and } N^-(v) = \{u \in V : uv \in A\}.$$

If $d(v) = k$ for every $v \in V$, then D is said to be a *k-regular* digraph. If for every $v \in V,\ d^-(v) = d^+(v)$, the digraph is said to be an *isograph* or a *balanced digraph*. We note that an isograph is an even degree digraph, but not necessarily regular. Also, every symmetric digraph is an isograph. Edmonds and Johnson [70] call isographs as asymmetric digraphs and Berge [23] calls them pseudo-symmetric graphs. Kotzig [137, 138] calls an anti-symmetric isograph as oriented-in-equilibrium or a ρ-graph.

A vertex v for which $d^+(v) = d^-(v) = 0$ is called an *isolate*. A vertex v is called a *transmitter* or a *receiver* according as $d^+(v) > 0,\ d^-(v) = 0$ or $d^+(v) = 0,\ d^-(v) > 0$. A vertex v is called a *carrier* if $d^+(v) = d^-(v) = 1$.

Underlying graph of a digraph: Let $D = (V,\ A)$ be a digraph. The graph $G = (V,\ E)$, where $uv \in E$ if and only if uv or vu or both are in A, is called the *underlying graph* of D. This is also called the *covering graph* $C(D)$ of D. Here we denote $C(D)$ by $G(D)$ or simply by G.

In case $G = (V,\ E)$ is a graph, the digraph with vertex set V and a symmetric uv whenever $uv \in E$, is called the *digraph corresponding* to G, and is denoted by $D(G)$, or D. Clearly, $D(G)$ is a symmetric digraph. An oriented graph obtained from the graph $G = (V,\ E)$ by replacing each edge $uv \in E$ by an arc uv or vu, but not both is called an *orientation* of G and is denoted by O(G) or O.

Complete symmetric digraph: A digraph $D = (V, A)$ is said to be *complete* if both uv and $vu \in A$, for all $u, v \in V$. Obviously this corresponds to K_n, where $|V| = n$, and is denoted by K_n^*. A complete antisymmetric digraph, or a complete oriented graph is called a *tournament.* Clearly, a tournament is an orientation of K_n (Fig. 11.2).

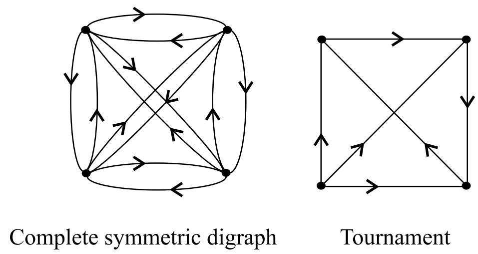


Fig. 11.2

We note that the number of arcs in K_n^* is $n\,(n-1)$ and the number of arcs in a tournament is $\frac{n(n-1)}{2}$.

11.2 Digraphs and Binary Relations

Let A and B be nonempty sets. A *(binary) relation R* from A to B is a subset of $A \times B$. If $R \subseteq A \times B$ and $(a, b) \in R$, where $a \in A$, $b \in B$, we say a "is related to" b by R, and we write aRb. If a is not related to b by R, we write $a \not R\, b$. A relation R defined on a set X is a subset of $X \times X$. For example, less than, greater than and equality are the relations in the set of real numbers. The property "is congruent to" defines a relation in the set of all triangles in a plane. Also, parallelism defines a relation in the set of all lines in a plane.

Let R define a relation on a nonempty set X. If R relates every element of X to itself, the relation R is said to be *reflexive*. A relation R is said to be *symmetric* if for all $x_i\, x_j \in X$, $x_i\, R\, x_j$ implies $x_j\, R\, x_i$. A relation R is said to be *transitive* if for any three elements x_i, x_j and x_k in X, $x_i R x_j$ and $x_j\, R\, x_k$ imply $x_i\, R\, x_k$. A binary relation is called an *equivalence relation* if it is reflexive, symmetric and transitive.

A binary relation R on a set X can always be represented by a digraph. In such a representation, each $x_i \in X$ is represented by a vertex x_i and whenever there is a relation R from x_i to x_j, an arc is drawn from x_i to x_j, for every pair (x_i, x_j). The digraph in Figure 11.3 represents the relation is less than, on a set consisting of four numbers 2, 3, 4, 6.

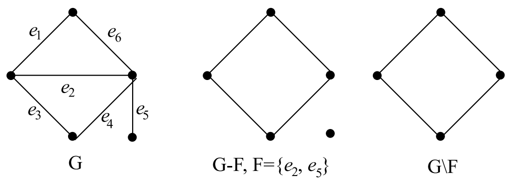


Fig. 11.3

We note that every binary relation on a finite set can be represented by a digraph without parallel edges and vice versa.

Clearly, the digraph of a reflexive relation contains a loop at every vertex (Fig. 11.4). A digraph representing a reflexive binary relation is called a reflexive digraph.

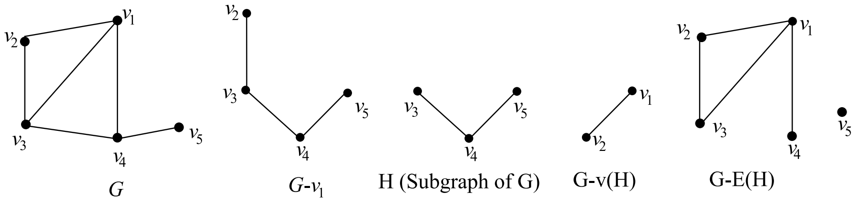


Fig 11.4

The digraph of a symmetric relation is a symmetric digraph because for every arc from x_i to x_j, there is an arc from x_j to x_i. Figure 11.5 shows the digraph of an irreflexive and symmetric relation on a set of three elements.

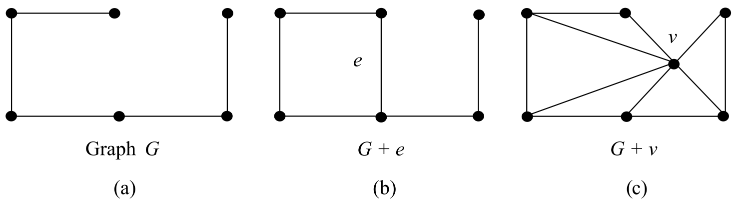


Fig. 11.5

A digraph representing a transitive relation on its vertex set is called a *transitive digraph*. Figure 11.6 shows the digraph of a transitive, which is neither reflexive, nor symmetric.

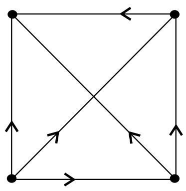


Fig. 11.6

A binary relation R on a set M can also be represented by a matrix, called a relation matrix. This is a (0, 1), $n \times n$ matrix $M_R = [m_{ij}]$, where n is the number of elements in M, and is defined by

$$m_{ij} = \begin{cases} 1 & if\ x_i\ R\ x_j\ is\ true, \\ 0, & otherwise. \end{cases}$$

Isomorphic digraphs: Two digraphs are said to be isomorphic if their underlying graphs are isomorphic and the direction of the corresponding arcs are same. Two non-isomorphic digraphs are shown in Figure 11.7.

Fig. 11.7

Subdigraph: Let $D = (V, A)$ be a digraph. A digraph $H = (U, B)$ is the *subdigraph* of D whenever $U \subseteq V$ and $B \subseteq A$. If $U = V$, the subdigraph is said to be *spanning*.

Complement of a digraph: The *complement* $\overline{D} = (V, \overline{A})$ of the digraph $D = (V, A)$ has vertex set V and $a \in \overline{A}$ if and only if $a \notin A$. That is, $\overline{D}$ is the relative complement of D in K_n^*, where $|V| = n$.

Converse digraph: The *converse* $D' = (V, A')$ of the digraph $D = (V, A)$ has vertex set V and $a = uv \in A'$ if and only if $a' = vu \in A$. That is, A' is obtained by reversing the direction of each arc of D. Clearly, $(D')' = D'' = D$.

A digraph D is *self-complementary* if $D \cong \overline{D}$ and D is said to be *self-converse* if $D \cong D'$. A digraph D is said to be *self-dual* if $D \cong \overline{D} \cong D'$.

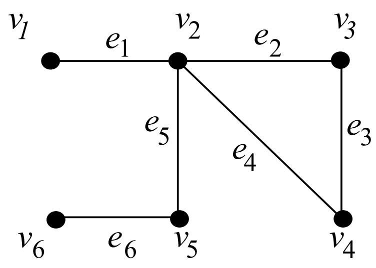
11.3 Directed Paths and Connectedness

Directed walks: A *(directed) walk* in a digraph $D = (V\ A)$ is a sequence $v_0\ a_1\ v_1\ a_2\ \ldots\ a_k\ v_k$, where $v_i \in V$ and $a_i \in A$ are such that $a_i = v_{i-1}\ v_i$ for $1 \le i \le k$, no arc being repeated. As there is only one arc of the form $v_i\ v_j$, the walk can also be represented by the vertex sequence $v_0 v_1 \ldots v_k$. A vertex may appear more than once in a walk. Clearly, the length of the walk is k. If $v_0 \neq v_k$, the walk is *open*, and if $v_0 = v_k$, the walk is *closed*. A walk is *spanning* if $V = \{v_0, \ldots, v_k\}$.

A *(directed) path* is an open walk in which no vertex is repeated. A *(directed) cycle* is a closed walk in which no vertex is repeated. A digraph is *acyclic* if it has no cycles.

A *semiwalk* is a sequence $v_0 a_1 v_1 a_2 \ldots a_k v_k$ with $v_i \in V$ and $a_i \in A$ such that either $a_i = v_{i-1} v_i$ or $a_i v_i v_{i-1}$ and no arc is repeated. The length of the semiwalk is k. If $v_0 \neq v_k$, the semiwalk is *open*, and if $v_o = v_k$, the semiwalk is *closed.* If no vertex is repeated in an open (closed) semiwalk, it is called a *semi path (semicycle).*

A spanning path of a digraph is called a Hamiltonian path and a spanning cycle is called a Hamiltonian cycle. A digraph with a Hamiltonian cycle is said to be Hamiltonian.



Graph G

Fig. 11.8

In Figure 11.8, $v_1\ a_1\ v_2\ a_2\ v_3\ a_4\ v_4$ is an open walk, $v_1\ a_1\ v_2\ a_3\ v_3\ a_4\ v_4$ is semiwalk, $v_1\ a_1\ v_2\ a_8\ v_4\ a_5\ v_5$ is a path and $v_1\ a_1\ v_2\ a_8\ v_4\ a_5\ v_5\ a_6\ v_1$ is a cycle.

In a digraph $D = (V, A)$, a vertex u is said to be *joined* to a vertex v, if there is a semipath from u to v. We note that the relation 'is joined to' is reflexive, symmetric and transitive, and therefore is an equivalence relation on V. A vertex u is said to be *reachable* from a vertex v, if there is a path from v to u. The relation 'is reachable from' is reflexive and transitive, but not symmetric, since there may or may not be a path from u to v. A vertex v is called a *source* of D if every vertex of D is reachable from v, and v is called a *sink* of D, if v is reachable from every other vertex.

Principle of duality for digraphs

While changing a digraph D to its converse D', we observe that the properties about D get changed to the corresponding properties about D'. When D' is changed to $D'' = D$, the original properties of D are obtained. Such type of a pair of properties are called *dual properties*, (transmitter, receiver), (source, sink), (indegree, outdegree), (isolate, isolate) and (carrier, carrier). The dual of a statement P about a digraph is the statement P' obtained from P by changing every concept in P to its dual. For any statements P and Q for digraphs, $P \Rightarrow Q$ in D is true if and only if $P' \Rightarrow Q'$ in D' and for any digraph D there is a converse D'. Therefore for every result in digraphs we get a dual result by changing every property to its dual. This is called the *principle of duality* for digraphs.

Definition: A digraph is said to be *strongly connected* or strong, if every two of its distinct vertices u and v are such that u is reachable from v and v is reachable from u. A digraph is *unilaterally connected* or unilateral, if either u is reachable from v or v is reachable from u and is *weakly connected* or weak, if u and v are joined by a semipath.

In Figure 11.9, (a) shows a strong digraph, (b) a unilateral digraph and (c) a weak digraph.

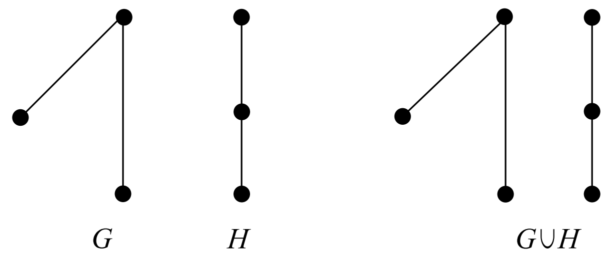


Fig. 11.9

A digraph is said to be *disconnected* if it is not even weak. A digraph is said to be *strictly weak* if it is weak, but not unilateral. It is *strictly unilateral*, if it is unilateral but not strong.

Two vertices of a digraph D are said to be

i. 0-connected if there is no semipath joining them,

ii. 1-connected if there is a semipath joining them, but there is no $u-v$ path or $v-u$ path,

iii. 2-connected if there is a $u-v$ or a $v-u$ path, but not both,

iv. 3-connected if there is $u-v$ path and a $v-u$ path.

Definition: An *arc sequence* in a digraph D is an alternating sequence of vertices and arcs of D.

The following results characterise various types of connectivity in digraphs.

Theorem 11.1 A digraph is strong if and only if it has a spanning closed arc sequence.

Proof

Necessity Let $D = (V, A)$ be a strong digraph with $V = \{v_1, v_2, \ldots, v_n\}$. Then there is an arc sequence from each vertex in V to every other vertex in V. Therefore, there exists in D, arc sequences $Q_1, Q_2, \ldots, Q_{n-1}$ such that the first vertex of Q_i is v_i and the last vertex of Q_i is v_{i+1}, for $i = 1, 2, \ldots, n-1$. Also, there exists an arc sequence, say Q_n , with first vertex v_n and the last vertex v_1. Then the arc sequence obtained by traversing the arc sequences $Q_1, Q_2, \ldots, Q_n$ in succession is a spanning closed arc sequence of D.

Sufficiency Let u and v be two distinct vertices of V. If v follows u in any spanning closed arc sequence, say Q of D, then there exists a sequence of the arcs of Q forming an arc sequence from u to v. If u follows v in Q, then there is an arc sequence from u to the last vertex of Q and an arc sequence from that vertex to v. An arc sequence from u to v is then obtained by traversing these two arc sequences in succession. ❑

Theorem 11.2 A digraph D is unilateral if and only if it has a spanning arc sequence.

Proof

Necessity Assume D is unilateral. Let Q be an arc sequence in D which contains maximum number of vertices and let Q begin at the vertex v_1 of D and end at the vertex v_2 of D. If Q is a spanning arc sequence, there is nothing to prove. Assume that Q is not a spanning arc sequence. Then there exists a vertex, say u of D that is not in Q. Also in D, there is neither an arc sequence from u to v_1, nor from v_2 to u. Since D is unilateral and does not contain an arc sequence from u to v_1, D contains an arc sequence from v_1 to u.

Let $w(\neq v_2)$ be the last vertex of Q from which an arc sequence from w to u exists in D. Let Q_1 be an arc sequence from w to u in D. Let z be the vertex in D which is the immediate successor of the last appearance of w in Q. Clearly, D does not contain an arc sequence from z to u. Since D is unilateral, there is an arc sequence, say Q_2 from u to z in D. Traversing Q from v_1 to the last appearance of w, then traversing u, then traversing Q_2 to vertex z and finally traversing Q to v_2, we obtain an arc sequence from v_1 to v_2 which has more distinct vertices than Q. This is a contradiction and thus Q is a spanning arc sequence in D (Fig. 11.10).

The sufficiency follows from the definition. ❑

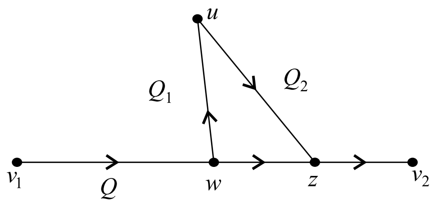


Fig. 11.10

Theorem 11.3 A digraph is weak if and only if it has a spanning semi arc sequence.

Proof Let $D = (V, A)$ be a weak digraph with $V = \{v_1, v_2, \ldots, v_n\}$. Since D is weak, there is a semi arc sequence, say Q_i from v_i to v_{i+1} in D for $i = 1, 2, \ldots, n-1$. The semi arc sequence obtained by traversing the semi arc sequences $Q_1, Q_2, \ldots, Q_n$ in succession is a spanning semi arc sequence of D.

Conversely, let D be a digraph containing a spanning semi arc sequence, say Q. Let v_1 and v_2 be two distinct vertices of D. Clearly, v_1 and v_2 are in Q, since Q is spanning. The part of Q which begins at any appearance of v_1 (v_2) and ends at any appearance of v_2 (v_1) represents a semi arc sequence from v_1 to v_2 (from v_2 to v_1) in D. Thus there is either a semi arc sequence from v_1 to v_2, or from v_2 to v_1 in D. Hence D is weak.

Definition: In a digraph D, a *strong component* is a maximal strong subdigraph of D. A *unilateral component* is a maximal unilateral subdigraph of D and a *weak component* is a maximal weak subdigraph of D. In Figure 11.1(b), the digraph has a strong component induced by the vertex set $\{v_2, v_3, v_4\}$. In Figure 11.1(c), the digraph has a unilateral component induced by the vertex set $\{v_1, v_4, v_2\}$ and a weak component which is the digraph itself.

Theorem 11.4

i. Every vertex and every arc of a digraph D belongs to a unique weak component.

ii. Every vertex and every arc of a digraph of D belongs to at least one unilateral component.

iii. Every vertex of a digraph belongs to a unique strong component. Every arc is contained in at most one strong component and it is in a strong component if and only if it is in a cycle.

Proof

i. If a vertex v lies in two weak components W_1 and W_2, let v_1 and v_2 be any two vertices of $W_1 - W_2$ and $W_2 - W_1$. Then there is a vv_1 and vv_2 semi-path. Also there is a v_1 v_2 semipath, so that W_1 and W_2 are in the same weak component. Similar argument holds for an arc.

ii. Since each vertex and each arc is a unilateral subdigraph, the result follows.

iii. Let v be any vertex and S be the set of vertices mutually reachable with v including v itself. Then $\langle$ S $\rangle$ is a strong component containing v. The uniqueness of this follows as in (i). Also, a similar argument holds for an arc.

If an arc uv is in a cycle, then all vertices on the cycle are pairwise reachable and belong to a strong component containing the arcs of the cycle, in particular uv. Conversely, if uv is in a strong component, u and v are mutually reachable and hence there is vu path, which together with uv gives a cycle. ❑

We note that the vertex sets of the weak components of a digraph give a partition π_w of its vertex set V and is called a *weak partition*. Similarly, the vertex sets of the strong components of a digraph give a partition π_s of V, which is a refinement of the partition π_w. This π_s is called the *strong partition*.

Definition: Let $V = S_1 \cup S_2 \cup \ldots \cup S_k$ be a partition π of the vertex set V of the digraph D = (V, A). Consider a digraph D_π with vertex set V_π = $\{S_1, S_2, \ldots, S_k\}$ which has an arc S_iS_j if and only if in D there is at least one arc from a vertex of S_i to a vertex of S_j. Then D_π is called the *contraction* of D with respect to the partition π.

The contraction of a digraph D with respect to its strong partition π_S is called the *condensation* D^* of D.

Definition: The symmetrisation D^S of a digraph D is obtained from D by adding uv to A whenever $uv \notin A$, but $vu \in A$. Equivalently, $D^S = D(G(D))$, which is the digraph corresponding to the underlying graph of D.

Figure 11.11 illustrates these operations.

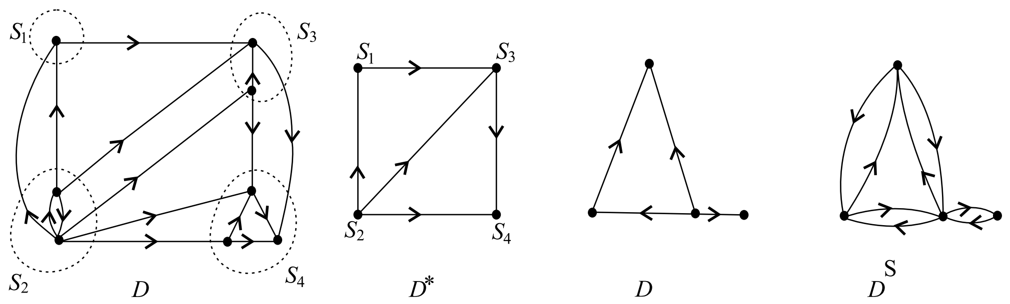


Fig. 11.11

We note from the definition of the condensation that a digraph D and its condensation have the same kind of connectedness.

Theorem 11.5 If S_1 and S_2 are two strong components of digraph D, and $v_1 \in S_1$ and $v_2 \in S_2$, then there is a $v_1 - v_2$ path in D if and only if there is an $S_1 - S_2$ path in D^*.

Proof Let there be a $v_1 - v_2$ path P in D. If length of P is one, then there is an $S_1 - S_2$ arc in D^*. We induct on the length of P. Let the result hold for any path of length $n-1$ in D. Assume that $P = v_1 u_1 u_2 \ldots u_{n-1} v_2$ is of length n. Let $S(u_{n-1})$ be the strong component containing u_{n-1}. Then by induction hypothesis, there is a path P^* from S_1 to $S(u_{n-1})$ in D^*. If $u_{n-1} \in S_2$, this path serves as an $S_1 - S_2$ path. If not, since $u_{n-1}v_2 \in A$, there is an arc from $S(u_{n-1})$ to S_2 in D^*, and thus an $S_1 - S_2$ path.

Conversely, let $S_1 S_3 S_4 \ldots S_2$ be an $S_1 - S_2$ path in D^*. Then there are arcs $u_1 u_3$, $u_3'\ u_4$, $u_4'\ u_5$, ..., $u_n' u_2$ in D, u_i, $u_i' \in S_i$. Since the S_i's are strong components, there are $v_1 - u_1$, $u_3 - u_3'$, $\ldots, u_n - u_n', u_2 - v_2$ paths in D. These along with the arcs given above form a $v_1 - v_2$ path in D. ❑

The following result can be easily established.

Theorem 11.6 If v_1, v_2 are vertices in different strong components S_1, S_2 of a digraph D, then there is a strict semipath but no path joining v_1 to v_2 if and only if there is a strict semipath and no path from S_1 to S_2 in D^*.

Now we have the following result.

Theorem 11.7 The condensation D^* of any digraph is acyclic.

Proof If D^* contains a cycle, let $S_1 S_2$ be an arc in this cycle. By Theorem 11.1, $S_1 S_2$ lies on a strong component of D^*. Therefore S_1 and S_2 are mutually reachable in D^*, and by Theorem 11.5, there are vertices $v_1 \in S_1$ and $v_2 \in S_2$ which are mutually reachable in D. Thus v_1 and v_2 belong to the same strong component S of D. So, $S_1 = S_2 = S$, and $S_1 S_2$ is not an arc in D^*, contradicting the assumption. ❑

We note that a digraph D is strong if and only if D^* consists of a single vertex.

Definition: A *cut-set* in a digraph $D = (V,A)$ is a set of arcs of A, which constitute a cut-set in the multigraph $G = (V,E)$, obtained from D by removing the orientation from each arc of A.

11.4 Euler Digraphs

A digraph D is said to be Eulerian if it contains a closed walk which traverses every arc of D exactly once. Such a walk is called an *Euler walk.* A digraph D is said to be *unicursal* if it contains an open Euler walk.

The following result characterises Eulerian digraphs.

Theorem 11.8 A digraph $D = (V,A)$ is Eulerian if and only if D is connected and for each of its vertices v, $d^-(v) = d^+(v)$.

Proof

Necessity Let D be an Eulerian digraph. Therefore, it contains an Eulerian walk, say W. In traversing W, every time a vertex v is encountered we pass along an arc incident towards v and then an arc incident away from v. This is true for all the vertices of W, including the initial vertex of W, say v, because we began W by traversing an arc incident away from v and ended W by traversing an arc incident towards v.

Sufficiency Let for every vertex v in D, $d^-(v) = d^+(v)$. For any arbitrary vertex v in D, we identify a walk, starting at v and traversing the arcs of D at most once each. This traversing is continued till it is impossible to traverse further. Since every vertex has the same number of arcs incident towards it as away from it, we can leave any vertex that we enter along the walk and the traversal then stops at v. Let the walk traversed so far be denoted by W. If W includes all arcs of A, then the result follows. If not, we remove from D all the arcs of W and consider the remainder of A. By assumption, each vertex in the remaining digraph, say D_1, is such that the number of arcs directed towards it equals the number of arcs directed away from it. Further, W and D_1 have a vertex, say u in common, since D is connected. Starting at u, we repeat the process of tracing a walk in D_1. If this walk does not contain all the arcs of D_1, the process is repeated until a closed walk that traverses each of the arcs of D exactly once is obtained. Hence D is Eulerian. ❑

Theorem 11.9 A weakly connected digraph $D = (V,A)$ is unicursal if and only if D contains vertices u and v such that $d^+(u) = d^-(u)+1, d^-(v) = d^+(v)+1$ and $d^+(w) = d^-(w)$, for all $w \in V$, where $w \neq u$, v. In this case, the open Euler walk begins at u and ends at v.

Proof Let D be unicursal. Then D has an Euler walk W that begins at u and ends at v. Therefore, as in Theorem 11.8, for every vertex w different from both u and v, we have

$d^+(w) = d^-(w)$. Also, the first arc of W contributes one to the outdegree of u while every other occurrence of u in W contributes one each to the outdegree of u. Thus, $d^+(u) = d^-(u)+1$.

Similarly, $d^-(v) = d^+(v)+1$.

Conversely, let D be a weakly connected digraph containing vertices u and v such that $d^+(u) = d^-(u)+1$, $d^-(v) = d^+(v)+1$ and for each $w \neq u,\ v$, $d^+(w) = d^-(w)$. In D, add a new arc a joining u and v. Now, we get a digraph D_1 in which the new outdegree of v is one more than its old outdegree, so that $d^-(v) = d^+(v)$. Similarly, $d^-(u) = d^+(u)$, and for every other vertex w, $d^-(w) = d^+(w)$. Also, D_1 is weakly connected, and since $d^-(z) = d^+(z)$ for every vertex z in D_1, it follows from Theorem 11.8 that D_1 is Eulerian. Let W be an Euler walk in D_1. Clearly, D_1 contains all the arcs of D together with the added arc a. Deleting the arc a produces an open Euler walk in D. Hence D is unicursal. ❑

Theorem 11.10 A non-trivial weak digraph is an isograph if and only if it is the union of arc-disjoint cycles.

Proof If the weak digraph D is a union of arc-disjoint cycles, each cycle contributes one to the indegree and one to the outdegree of each vertex on it. Thus, $d^+(v) = d^-(v)$, for all $v \in V$.

Conversely, let D be a non-trivial weak isograph. Then each vertex has positive outdegree and therefore D has a cycle, say Z. Removing the edges of Z from D, we get a digraph D_1 whose weak components are isographs. By using an induction argument, each such non-trivial weak component is a union of arc-disjoint cycles. These cycles together with Z provide a decomposition of the arc set of D into cycles. ❑

Corollary 11.1 Every weak isograph is strong.

Proof If u and v are any two vertices of the weak isograph D, there is a semi path P joining u and v, and each arc of this lies on some cycle of D. The union of these cycles provides a closed walk containing u and v. Thus u and v are mutually reachable. ❑

11.5 Hamiltonian digraphs

Definition: A spanning path of a digraph is called a Hamiltonian path and a spanning cycle a Hamiltonian cycle. A digraph containing a Hamiltonian cycle is said to be Hamiltonian.

Regarding the results on Hamiltonian digraphs, there are surveys by Bermond and Thomassen [24], and by Jackson. There are a number of results that are analogous to those proved for Hamiltonian graphs in Chapter 3. Two early results on sufficient conditions are due to Ghouila-Houri [87] and Woodall [270]. But now we have a more general result due to Meyniel [160]. The proof of Meyniel's result given here is due to Bondy and Thomassen [38].

First, we have the following observations.

Lemma 11.1 Let $P = v_1 v_2 \ldots v_k$ be a path in the digraph D and v be any vertex in $V - V(P)$. If there is no $v_1 - v_k$ path with vertex set $V(P) \cup \{v\}$, then $|\{v,\ V(P)\}| \leq k+1$, where $\{v,\ V(P)\}$ is the set of all arcs in D with one end in v or $V(P)$ and the other end in $V(P)$ or v, respectively.

Proof By the assumption on P, for any $v \in V - V(P)$, there is no path $v_i\ v\ v_{i+1}$ in D. Therefore for each i, $1 \leq i \leq k-1$, $|(v_i,\ v)| + |(v,\ v_{i+1})| \leq 1$.

$$\text{Hence, } |\{v,\ V(P)\}| = \sum_{i=1}^{k-1} (|v_i\ , v)| + |v,\ v_{i+1}| + |(v,\ v_1)| + |(v_k,\ v)|$$

$$\leq (k-1)1 + 2 = k+1. \qquad \square$$

Let in a digraph $D = (V,\ A)$, S be a proper subset of V. A $u - v$ path of length at least two with only u and v in S is called an *S-path*.

Theorem 11.11 Let D be a strong non-Hamiltonian digraph and $C = v_1 v_2 \ldots v_k v_1$ be a cycle of D such that there is no cycle of D whose vertex set properly contains $V(C) = C$, say. Then there exists a $v \in V - C$, and integers p and $q(1 \leq p,\ q \leq k)$ such that (i) $v_p\, v \in A$, (ii) $vv_{p+i} \notin A$ for i, $1 \leq i \leq q$ and (iii) $d(v) + d(v_{p+q}) \leq 2n - 1 - q$.

Proof

Case 1 D has no C-path. Since D is strong and C is a proper subset of V, there is a vertex in C joined to some vertex of $V - C$ by a path and joined from some vertex of $V - C$ by a path. By the assumption on C, there is a cycle C' having only one vertex, say v_p, common with C. Let v be the successor of v_p on C' (Fig. 11.12(a)). If v is adjacent to or from any vertex of C other than v_p, D has a C-path. Therefore, $|\{v, C\}| \leq 2$. Clearly, $|\{v_{p+1}, C\}| \leq 2(k-1)$. For any $u \in V - C$, if vuv_{p+1} or $v_{p+1}uv$ is a 2-path, then these are C-paths in D. Thus, such adjacencies are not possible. Therefore, $|\{u, \{v,\ v_{p+1}\}\}| \leq 2$ for such u. Putting these together, we have

$$d(v) + d(v_{p+1}) \leq 2 + 2(k-1) + 2(n-k-1) = 2n-2,$$

and this verifies the statements with $q = 1$.

Case 2 D has a C-path, say $P = v_p\ u_1 u_2 \ldots u_s v_p$, where $u_i \in V - C$. Choose P such that r is least. By the assumption on C, $r > 1$. Let $v = u_1$.

i. By the assumption on C and the minimality of r, v is not adjacent to any v_{p+i}, $1 \leq i \leq r-1$. Therefore the path $Q = v_{p+r}\, v_{p+r+1} \ldots v_k \ldots v_1 \ldots v_p$ and v satisfy the hypothesis of Lemma 11.1 and we have

$$|\{v,\ V(Q)\}| = |\{v,\ C\}| \leq k - r + 2. \qquad (11.11.1)$$

ii. By the minimality of r, we observe that for any $u \in V - C$, vuv_{p+i} or $v_{p+i}\ uv$ are not possible paths in D, for $1 \le i \le r-1$ (Fig. 11.12(b)). Thus, for any $u \in V - C$,

$$|\{u,\ \{v,\ vp+i\}\}| = 2, \text{ for any such } i. \tag{11.11.2}$$

iii. Since D is strong, there are $v_p - v_{p+r}$ and $v_{p+r} - v_p$ paths in D. Clearly, $v_{p+r}v_{p+r+1}\ldots v_k v_1 v_2 \ldots v_p$ is such a $v_{p+r} - v_p$ path. It is possible that there are other $v_{p+r} - v_p$ paths containing all these vertices and some more, say $v_{p+1}\ \ v_{p+2}\ldots . v_{p+i-1}$. Let q be the largest integer i, $1 \le i \le r$ such that there is a $v_{p+r} - v_p$ path with vertex set $S = \{v_{p+r}, v_{p+r+1}, \ldots, v_1, v_2, \ldots, v_{p-1}, v_{p+1}, \ldots, v_{p+q-2}, v_{p+q-1}, v_p\}$ and let P' be such a path. By the assumption on C, S cannot contain all the vertices of C, since $P \cup P'$ is a cycle. Therefore, $q < r$. It is possible that $q = 1$ (Fig. 11.12 (c)). Now the lemma is applicable for v_{p+q} and P'. Therefore,

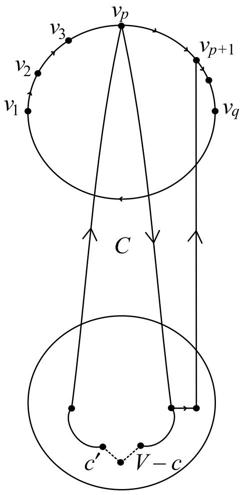
$$|\{V_{p+q}, V(P')\}| \le k - r + q + 1. \tag{11.11.3}$$

Using (11.11.2) with $i = q$, we have

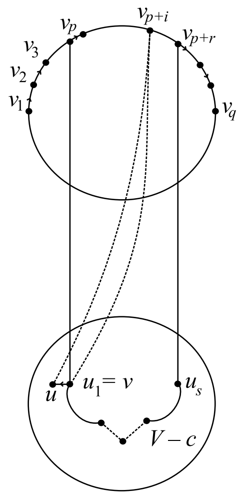
$$|\{u, \{v, v_{p+q}\}\}| \le 2, \text{ for each } u \in V - C. \tag{11.11.4}$$

Also, v_{p+q} can be joined to and from any of the other $r - q - 1$ vertices among

$$\{v_{p+q+1}, \ldots, v_{p+r-1}\}. \tag{11.11.5}$$



(a)



(b)

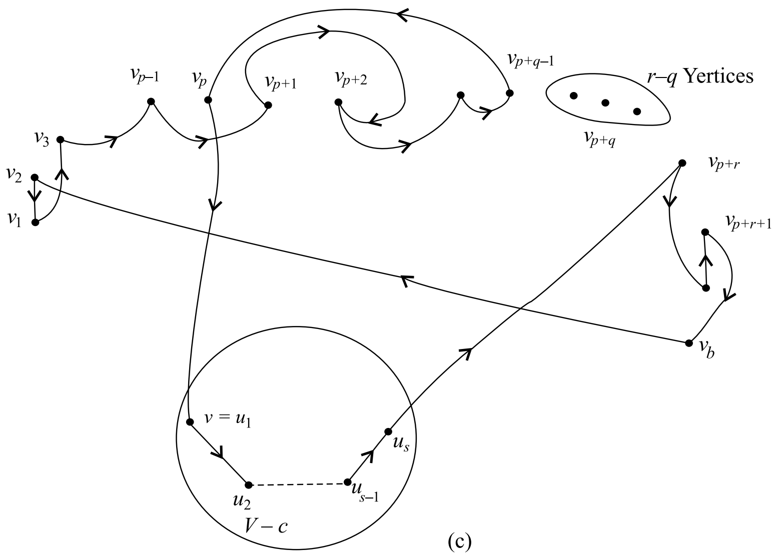


Fig. 11.12

Combining all these, we get

$$\begin{aligned} d(v)+d(v_{p+q}) &= |\{v,C\}|+|\{v_{p+q},V(P')\}|+|\{v_{p+q},\{v_{p+q+1},\ldots,v_{p+r-1}\}\}| \\ &\quad + \sum_{u\in V-C} |\{u,\{v,v_{p+q}\}\}| \\ &\le (k-r+2)+(k-r+q+1)+2(r-q-1)+2(n-k-1) \\ &= 2n-q-1. \end{aligned}$$

❑

Theorem 11.12 (Meyneil) If D is a strong digraph of order n such that for any pair of non-adjacent vertices u and v, $d(u)+d(v) \ge 2n-1$, then D is Hamiltonian.

Proof If such a D is non-Hamiltonian, by Theorem 11.11, there exists a pair of non-adjacent vertices v and v_{p+q} such that $d(v)+d(v_{p+q}) < 2n-1$, contradicting the hypothesis. ❑

Corollary 11.2 If D is a strong digraph such that for any vertex v, $d(v) = d^+(v)+d^-(v) \ge n$, then D is Hamiltonian.

The direct proof of this result can be found in Berge [18].

11.6 Trees with Directed Edges

We know that a tree in undirected graphs is a connected graph without cycles. But in case of digraphs, a structure similar to that of a tree needs absence of cycles as well as absence of semi cycles. We have the following definition.

A *(directed) tree* is a connected digraph without cycles, neither directed cycles nor semi cycles. We observe that a tree with n vertices has $n-1$ directed edges, and has properties analogous to those of trees with undirected edges. A digraph whose weak components are trees is called a *forest*. For example, Figure 11.13 shows a tree.

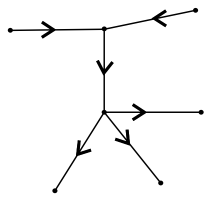


Fig. 11.13

Arborescence: A (directed) tree is said to be an *arborescence* if it contains exactly one vertex, called the *root*, with no arcs directed towards it and if all the arcs on any semipath are directed away from the root. For example, the tree in Figure 11.14 is an arborescence. That is, every vertex other than the root has indegree exactly one. Arborescence is also called an *out-tree*. If the direction of every arc in an arborescence is reversed, we get a tree called an *in-tree*.

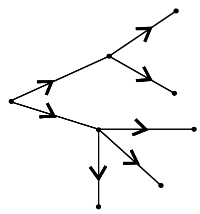


Fig. 11.14

Theorem 11.13 In an arborescence, there is a directed path from the root v to every other vertex. Conversely, a digraph D without cycles is an arborescence if there is a vertex v in D such that every other vertex is reachable from v and v is not reachable from any other vertex.

Proof In an arborescence, consider a directed path P starting from the root v and continuing as far as possible. Clearly, P can end only at a pendant vertex, since otherwise, we get a vertex whose indegree is two or more, which is a contradiction. Since an arborescence is connected, every vertex lies on some directed path from the root v to each of the pendant vertices.

Conversely, since every vertex in D is reachable from v and D has no cycle, D is a tree. Further, since v is not reachable from any other vertex, $d^{-}(v) = 0$. Every other vertex is reachable from v and therefore indegree of each of these vertices is at least one. The indegree is not greater than one, because there are only $n - 1$ arcs in D, n being the number of vertices of D. ❑

Ordered trees: A tree in which the relative order of subtrees meeting at each vertex is preserved is called an *ordered tree* or a planar tree (because the tree can be visualised as rigidly embedded in the plane of the paper). In computer science, the term tree usually means an ordered tree and by convention, a tree is drawn hanging down with the root at the top.

Spanning trees: A *spanning tree* is an n-vertex connected digraph analogous to a spanning tree in an undirected graph and consists of $n - 1$ directed arcs. A *spanning arborescence* in a connected digraph is a spanning tree that is an arborescence. For example, $\{a, b, c, g\}$ is a spanning arborescence in Figure 11.15.

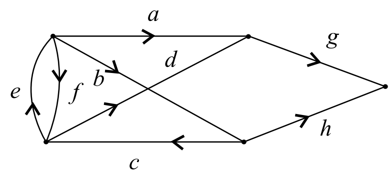


Fig. 11.15

Theorem 11.14 In a connected isograph D of n vertices and m arcs, let $W = (a_1, a_2, \ldots, a_m)$ be an Euler line, which starts and ends at a vertex v (that is, v is the initial vertex of a_1 and the terminal vertex of a_m). Among the m arcs in W there are $n - 1$ arcs that enter each of $n - 1$ vertices, other than v, for the first time. The subdigraph D_1 of these $n - 1$ arcs together with the n vertices is a spanning arborescence of D, rooted at vertex v.

Proof In the subdigraph D_1, vertex v is of indegree zero, and every other vertex is of indegree one, for D_1 includes exactly one arc going to each of the $n - 1$ vertices and no arc going to v. Further, the way D_1 is defined in W, implies that D_1 is connected and contains $n - 1$ arcs. Therefore D_1 is a spanning arborescence in D and is rooted at v. ❑

Illustration: In Figure 11.16, $W = (b\ d\ c\ e\ f\ g\ h\ a)$ is an Euler line, starting and ending at vertex 2. The subdigraph $\{b, d, f\}$ is a spanning arborescence rooted at vertex 2.

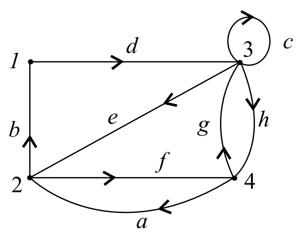


Fig. 11.16

The following result is due to Van Aardenne-Ehrenfest and N.G. de Bruijn [257].

Theorem 11.15 Let D be an Euler digraph and T be a spanning in-tree in D, rooted at a vertex v. Let a_1 be an arc in D incident out of the vertex v. Then a directed walk $W = (a_1, a_2, \ldots, a_m)$ is a directed Euler line, if it is constructed as follows.

i. No arc is included in W more than once.

ii. In exiting a vertex the one arc belonging to T is not used until all other outgoing arcs have been traversed.

iii. The walk is terminated only when a vertex is reached from which there is no arc left on which to exit.

Proof The walk W terminates at v, since all vertices have been entered as often as they have been left (because D is an isograph). Now assume there is an arc a in D that has not been included in W. Let u be the terminal vertex of a. Since D is an isograph, u is also the initial vertex of some arc b not included in W. Arc b going out of vertex u is in T, according to (i). This omitted arc leads to another omitted arc c in T, and so on. Finally we arrive at v and find an outgoing arc not included in W. This contradicts (iii). ❑

Theorem 11.14 provides a method of obtaining a spanning arborescence rooted at any specified vertex, provided the digraph is Eulerian. Conversely, given a spanning arborescence in an Euler digraph, an Euler line can be constructed using Theorem 11.15.

The number of distinct Euler lines formed from a given in-tree T and starting with arc a_1 at v, can be computed by considering all the choices available at each vertex, after starting with a_1. Since there is exactly one outgoing arc in T at each vertex and this arc is to be selected last ((ii), Theorem 11.15), the remaining $d^+(v_i) - 1$ arcs at vertex v_i can be chosen in $(d^+(v_i) - 1)!$ ways. Since these are independent choices, we have altogether $\prod_{i=1}^{n} (d^+(v_i) - 1)!$ different Euler lines that meet (i), (ii) and (iii) of Theorem 11.15.

Illustration: Consider Figure 11.17. We apply (i), (ii) and (iii) of Theorem 11.15 to obtain different Euler lines from the in-tree $\{a_2, a_3, a_7, a_{10}, a_{11}\}$, starting with arc a_1. The two Euler lines obtained are $(a_1\, a_{12}\, a_5\, a_6\, a_7\, a_8\, a_9\, a_{10}\, a_{11}\, a_2\, a_4\, a_3)$ and $(a_1\, a_{12}\, a_8\, a_9\, a_{10}\, a_{11}\, a_5\, a_6\, a_7\, a_2\, a_4\, a_3)$.

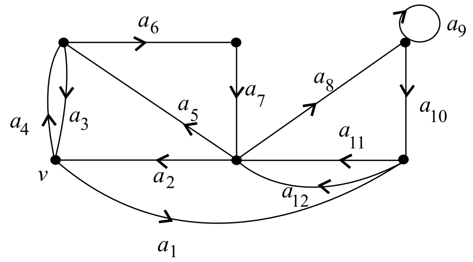


Fig. 11.17

Here, $\prod_{i=1}^{n} (d^+(v_i) - 1)! = 2.$

Note that these are not all the Euler lines in the digraph, but only those that are generated by the specific in-tree in accordance with (i), (ii) and (iii) of Theorem 11.15.

Fundamental cycles in digraphs:
The arcs of a connected digraph not included in a specified spanning tree T are called the *chords* with respect to T. As in undirected graphs, every chord c_i added to the spanning tree T produces a fundamental cycle, which is a directed cycle or a semi cycle.

A *cut-set* in a connected digraph D induces a partitioning of the vertices of D into two disjoint subsets V_1 and V_2 such that the cut-set consists of all those arcs that have one end vertex in V_1 and the other in V_2. All arcs in the cut-set can be directed from V_1 to V_2, or from V_2 to V_1, or some arcs can be directed from V_1 to V_2 and others from V_2 to V_1. A cut-set in which all arcs are oriented in the same direction is called a *directed cut-set*.

Consider the digraph of Figure 11.18. A spanning tree $T = \{a, d, f, h, k\}$ is shown by bold lines. Here, rank $= 5$, nullity $= 4$. The chord set with respect to T is $\{b, c, e, g\}$. Fundamental cycles with respect to T are dfe (semi cycle), $dkhc$ (semi cycle), khg (semi cycle) and $adkhb$ (directed cycle). The fundamental cut-sets with respect to T are ab, $bcde$, ef, $bcgk$ and $bcgh$.

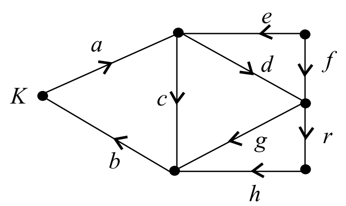


Fig. 11.18

11.7 Matrices A, B and C of Digraphs

The matrices associated with a digraph are almost similar to those discussed for an undirected graph, with the difference that in matrices of digraphs consist of 1, 0, -1 instead of only 0 and 1 for undirected graphs. The numbers 1, 0, -1 are real numbers and their addition and multiplication are interpreted as in ordinary arithmetic, not modulo 2 arithmetic as in undirected graphs. Thus the vectors and vector spaces associated with a digraph and its subdigraphs are over the field of all real numbers, but not modulo 2.

Incidence matrix:
The incidence matrix of a digraph with n vertices, m arcs and no self-loops is an $n \times m$ matrix $A = [a_{ij}]$, whose rows correspond to vertices and columns correspond to arcs, such that

$$a_{ij} = \begin{cases} 1, & if\ jth\ arc\ is\ incident\ out\ of\ ith\, vertex\ , \\ -1, & if\ jth\ arc\ is\ incident\ into\ ith\ vertex\ , \\ 0, & if\ jth\ arc\ is\ not\ incident\ on\ ith\ vertex\ . \end{cases}$$

For example, consider the digraph of Figure 11.19.

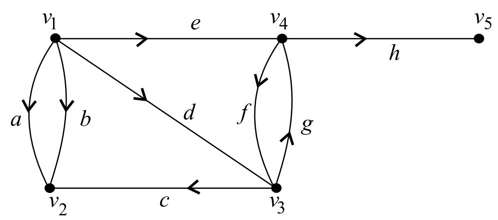


Fig. 11.19

The incidence matrix is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

Now, since the sum of each column is zero, the rank of the incidence matrix of a digraph of n vertices is less than n. The proof of the following result is almost similar to the result in undirected graphs.

Theorem 11.16 If $A(D)$ is the incidence matrix of a connected digraph of n vertices, then rank of $A(D) = n-1$.

We further note that after deleting any row from A, we get A_f, the $(n-1)\times m$ reduced incidence matrix. The vertex corresponding to the deleted row is called the reference vertex.

We now have the following result.

Theorem 11.17 The determinant of every square submatrix of A, which is the incidence matrix of a digraph, is -1, or 1, or 0.

Proof Consider a $k \times k$ submatrix M of A. If M has any column or row consisting of all zeros, then clearly det $M = 0$. Also, det $M = 0$, if every column of M contains the two non-zero entries, 1 and -1.

Now let $\det M \neq 0$. Then the sum of entries in each column of M is not zero. Therefore M has a column in which there is a single non-zero element that is either 1 or -1. Let this single element be in the (i, j)th position in M. Thus,

$$\det M = \pm 1 \det M_{ij},$$

where M_{ij} is the submatrix of M with its ith row and jth column deleted. The $(k-1) \times (k-1)$ submatrix M_{ij} is non-singular (because M is non-singular), therefore M_{ij} also has at least one column with a single non-zero entry, say in the (p, q)th position. Expanding det M_{ij} about this element in the (p, q)th position, we obtain

$$\det M_{ij} = \pm \text{ [det of non-singular } (k-2) \times (k-2) \text{ submatrix of } M].$$

Repeated application of this procedure gives

$$\det M = \pm 1.$$ ❑

Unimodular matrix: A matrix is said to be unimodular if the determinant of its every square submatrix is 1, -1, or 0.

Cycle matrix of a digraph: Let D be a digraph with m arcs and q cycles (directed cycles or semi cycles). An arbitrary orientation (clockwise or counter clockwise) is assigned to each of the q cycles. Then a cycle matrix $B = [b_{ij}]$ of the digraph G is a $q \times m$ matrix defined by

$$b_{ij} = \begin{cases} 1, & \textit{if the ith cycle includes the jth arc, and the orientations of the arc} \\ & \textit{and cycle coincide}, \\ -1, & \textit{if the ith cycle includes the jth arc, and the orientations of the two} \\ & \textit{are opposite}, \\ 0, & \textit{if the ith cycle does not include the jth arc}. \end{cases}$$

Example Consider the digraph D given in Figure 11.20. The cycle matrix of D is

$$B = \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

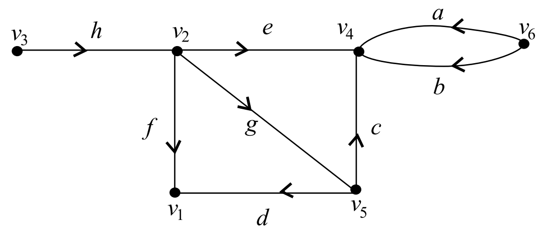


Fig. 11.20

We note that the orientation assigned to each of the four cycles is arbitrary. The cycle in the first row is assigned clockwise orientation, in the second row counter-clockwise, in the third counter-clockwise, and in the fourth clockwise. Changing the orientation of any cycle will simply change the sign of every non-zero entry in the corresponding row. Also we observe that if first row is subtracted from second, the third is obtained. Thus the rows are not all linearly independent (in the real field).

The next result gives the relation between incidence matrix and cycle matrix.

Theorem 11.18 Let B and A be respectively, the cycle matrix and the incidence matrix of a digraph (without loops) such that the columns are arranged using the same order of arcs. Then,

$$AB^T = BA^T = 0,$$

where T denotes the transposed matrix.

Proof Consider the pth row of A and the rth row of B. The rth cycle, say Z_r, either (a) does not, or (b) does possess an arc incident with vertex, say v_p, represented by the pth row of A. If (a), the product of the two rows is zero. If (b), there are exactly two arcs, say a_i and a_j, of the rth cycle incident with v_p.

We have the following four possibilities.

i. a_i and a_j are both incident towards v_p,

ii. a_i and a_j are both incident away from v_p,

iii. the directions of both a_i and a_j coincide with the orientation of Z_r and

iv. the directions of both a_i and a_j do not coincide with the orientation of Z_r.

It can be easily verified that in all these four cases, the product of the pth row of A and the rth row of B is zero. ❑

Figure 11.21 illustrates these four cases.

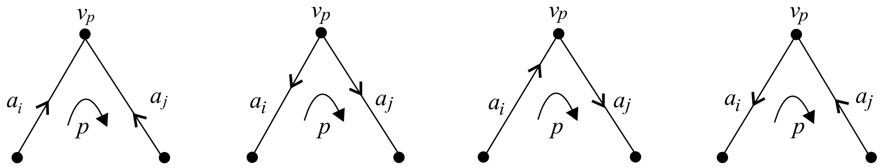


Fig. 11.21

Now, using Sylvester's theorem and Theorem 11.18, we can show that

rank $B+$ rank $A = m$.

If the digraph is connected, then rank $A = n-1$.

Therefore, rank $B = m-n+1$.

The following two results can be easily established.

Theorem 11.19 The non-singular submatrices of order $n-1$ of A are in one-one correspondence with spanning trees of a connected digraph of n vertices.

Theorem 11.20 The non-singular submatrices of B of order $\mu = m-n+1$ are in one-one correspondence with the chord set (complement of the spanning tree) of the connected digraph of n vertices and m edges.

Sign of a spanning tree: For a digraph, the determinant of the non-singular submatrix of A corresponding to a spanning tree T has a value either 1 or -1. This is referred to as the sign of T.

We note that the sign of a spanned tree is defined only for a particular ordering of vertices and arcs in A, because interchanging two rows or columns in a matrix changes the sign of its determinant. Thus the sign of a spanning tree is relative. Once the sign of one spanning tree is arbitrarily chosen, the sign of every other spanning tree is determined as positive or negative with respect to this spanning tree.

Number of spanning trees: The following result determines the number of spanning trees in a connected digraph.

Theorem 11.21 If A_f is the reduced incidence matrix of a connected digraph, then the number of spanning trees in the graph is equal to det $(A_f \, . \, A_f^T)$.

Proof According to Binet-Cauchy theorem,

$$\det(A_f \,.\, A_f^T) = \text{sum of the products of all corresponding majors of } A_f \text{ and } A_f^T.$$

Every major of A_f or A_f^T is zero unless it corresponds to a spanning tree, in which case its value is 1 or $-$ 1. Since both majors of A_f and A_f^T have the same value 1 or -1, the product is 1 for each spanning tree. ❑

Fundamental cycle matrix: The μ fundamental cycles each formed by a chord with respect to some specified spanning tree, define a fundamental cycle matrix B_f for a digraph. The orientation assigned to each of the fundamental cycles is chosen to coincide with that of the chord. Therefore B_f, a $\mu \times m$ matrix can be expressed exactly in the same form as in the case of an undirected graph,

$$B_f = [I_\mu : B_t],$$

where I_μ is the identity matrix of order μ and the columns of B_t correspond to the arcs in a spanning tree. This is illustrated in Figure 11.22.

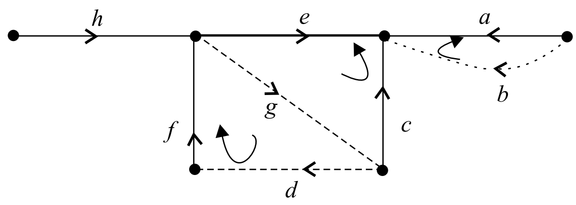


Fig. 11.22

$$\text{Here } B_f = \begin{array}{c} \begin{matrix} b & d & g & a & c & e & f & h \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array} = [I_\mu : B_t].$$

Cut-set matrix: Let $D = (V, A)$ be a connected digraph with q cut-sets. The cut-set matrix $C = [c_{ij}]$ of D is a $q \times m$ matrix in which the rows correspond to the cut-sets of D and the columns to the arcs of D. Each cut-set is given an arbitrary orientation. Let R_i be the ith cut-set of D and let R_i partition V into nonempty vertex sets V_i' and V_i''. The orientation can be defined to be either from V_i' to V_i'' or from V_i'' to V_i'. Suppose that the orientation is chosen to be from V_i' to V_i''. Then the orientation of an arc a_j of cut-set R_i is said to be the same as that of R_i if a_j is of the form $v_a\, v_b$, where $v_a \in V_i'$ and $v_b \in V_i''$ and opposite, otherwise. Then,

$$c_{ij} = \begin{cases} 1, & if\ arc\ a_j\ of\ cut-set\ R_i\ has\ the\ same\ orientation\ as\ R_i\,, \\ -1, & if\ arc\ a_j\ has\ the\ opposite\ orientation\ to\ R_i, \\ 0, & otherwise\,. \end{cases}$$

We have the following observations.

1. A permutation of the rows or columns corresponds to a relabelling of the cut-sets and arcs of D respectively.
2. Rank $C \geq$ rank A.
3. Rank $C \geq n-1$, by observation 2.
4. If the arcs of D are arranged in the same column order in B and C, then $BC^T = CB^T = 0$.
5. Rank $B+$ rank $C \leq m$.
6. If D is weak, rank $B = m-n+1$ and rank $C \leq n-1$.
7. Rank $C = n-1$, because of (3) and (4).

We observe that the removal of an arc, say $a = v_s v_t$ (also called a branch) of a spanning directed tree of D, partitions the vertices of a digraph D into two disjoint sets, say V_1 and V_2.

The cut-set created by the removal of a is said to be either (i) directed away from V_1 and towards V_2 if $v_s \in V_1$ and $v_t \in V_2$, or (ii) directed away from V_1 and towards V_2 if $v_s \in V_2$ and $v_t \in V_1$.

This type of cut-set is called *fundamental cut-set*. Clearly, not all the chords in R_i necessarily have the same orientation as $v_s v_t$. If $v_s v_t$ is directed away from a vertex in V_1, there may exist a chord in R_i which is directed towards a vertex in V_1. The orientation of a cut-set on the basis of the direction of the branch giving rise to it constitutes a natural way of orienting cut-sets. If all the chords of R_i are oriented as is $v_s v_t$, then R_i is said to be *directed*. Consider the graph shown in Figure 11.23 with T shown by bold lines. The fundamental cut-sets with respect to T are

branch	cut-set
a_3	$\{a_3, a_2\}$
a_5	$\{a_5, a_4\}$
a_6	$\{a_6, a_1, a_4\}$
a_7	$\{a_7, a_1, a_2\}$

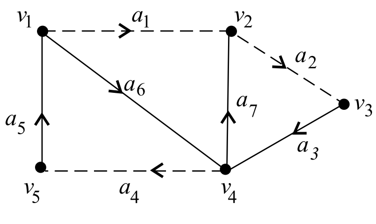


Fig. 11.23

A fundamental cut-set is created from C, the cut-set matrix of a connected digraph with the given directed spanning tree T, by deleting from C, all rows which do not correspond to fundamental cut-sets with respect to T. Therefore C_f is an $(n-1) \times m$ submatrix of C such that each row represents a unique fundamental cut-set with respect to T.

The rows of any fundamental cut-set C_f can be permuted to create a matrix of the form $C_f = [C_c : I_{n-1}]$, where C_c is an $(n-1) \times (m-n+1)$ matrix whose columns correspond to the chords of T and I_{n-1} is the identity matrix of order $n-1$ whose columns correspond to the branches of T.

Relation between $\mathbf{B}_f$, $\mathbf{C}_f$ and $\mathbf{A}_r$ A_r is the reduced incidence matrix in which an arbitrary row has been removed in order to make its rows linearly independent.

We have $B_f = [I_\mu : B_t]$ (11.7.i)

and $C_f = [C_c : I_{n-1}]$, (11.7.ii)

where t corresponds to the branches of T and c to the chords of T. Let the arcs be arranged in the same order in (11.7.i) and (11.7.ii) and in A_r. Partition A_r as

$$A_r = [A_c : A_t],$$

where A_c is an $(n-1) \times (m-n+1)$ submatrix whose columns correspond to the chords of T and A_t is an $(n-1) \times (n-1)$ submatrix whose columns correspond to the branches of T.

Since $AB^T = 0$, therefore $A_r\, B_f^T = 0$.

Thus, $[A_c : A_t] \begin{bmatrix} I_\mu \\ .. \\ B_t^T \end{bmatrix} = 0$, so that $A_c + A_t B_t^T = 0$.

Since A_t is non-singular, we have $A_t^{-1}[A_c + A_t B_t^T] = 0$.

Therefore, $A_t^{-1}A_c + B_t^T = 0$ and so $A_t^{-1}A_c = -B_t^T$. Also, $C_f B_f^T = 0$.

Therefore, $[C_c : I_{n-1}] \begin{bmatrix} I_\mu \\ .. \\ B_t^T \end{bmatrix} = 0$, and so $C_c + B_t^T = 0$.

Thus, $C_c = -B_t^T$ and so $C_c = A_t^{-1}A_c$.

Example Consider the digraph given in Figure 11.24, with spanning tree shown in bold lines.

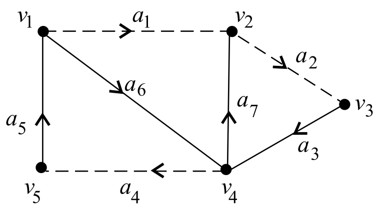


Fig. 11.24

We have

$$A_r = [A_c : At] = \begin{array}{c} \begin{array}{ccccccccc} a_1 & a_4 & a_2 & & a_5 & a_6 & a_7 & a_3 \end{array} \\ \left[\begin{array}{ccccccccc} -1 & 0 & 0 & \vdots & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & \vdots & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & \vdots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & & 0 & -1 & 1 & -1 \end{array}\right] \end{array}$$

$$B_f = [I_3 : B_t] = \begin{array}{c} \begin{array}{cccccccc} a_1 & a_4 & a_2 & & a_5 & a_6 & a_7 & a_3 \end{array} \\ \left[\begin{array}{cccccccc} 1 & 0 & 0 & \vdots & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & \vdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

$$C_f = [C_c : I_4] = \begin{array}{c} \begin{array}{cccccccc} a_1 & a_4 & a_2 & & a_5 & a_6 & a_7 & a_3 \end{array} \\ \left[\begin{array}{cccccccc} 0 & -1 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & \vdots & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Note that the last row of A, corresponding to vertex v_5, has been removed to form A_f. We form linear combinations of the rows of C_f to create 10 rows of C, representing all of the cut-sets of the digraph in Figure 11.24.

$$C = \begin{array}{cc} \begin{array}{ccccccc} a_1 & a_4 & a_2 & a_5 & a_6 & a_7 & a_3 \end{array} & \\ \left[\begin{array}{ccccccc} 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_1 - c_2 \\ c_2 - c_3 + c_4 \\ c_2 - c_1 - c_3 \\ c_1 - c_2 + c_3 - c_4 \\ c_2 - c_3 \\ c_3 - c_4 \end{array} \end{array}$$

The above facts lead to the following observations.

1. Given A_r, we can construct B_f and C_f.
2. Given B_r, we can construct C_f.

3. Given C_f, we can construct B_f.

Semipath matrix: The semipath matrix $P(u, v) = [p_{ij}]$, of a digraph $D = (V, A)$, where $u, v \in V$, is the matrix with each row representing a distinct semipath from u to v and the columns representing the arcs of D, in which $p_{ij} = 1$, if the ith semipath contains the jth arc, $p_{ij} = -1$ if the ith semipath contains the converse of the jth arc, and $p_{ij} = 0$ otherwise.

The matrix $P(v_3, v_5)$ for the digraph of Figure 11.25 is

$$P(v_3, v_5) = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

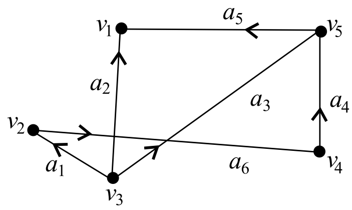


Fig. 11.25

We have the following observations about P.

1. If $P(u, v)$ contains a column of all zeros, then the vertex that it represents does not belong to any of the semipaths between u and v.
2. If $P(u, v)$ contains a column of all unit entries, then the vertex that it represents belongs to every semipath between u and v.
3. The number of non-zero entries in any row of $P(u, v)$ equals the number of arcs in the semipath represented by the row.

Adjacency matrix of a digraph: Let G be a digraph with n vertices and with no parallel arcs. The adjacency matrix $X = [x_{ij}]$ of the digraph G is an $n \times n$ (0, 1)-matrix defined by

$$x_{ij} = \begin{cases} 1, & \textit{if there is an arc directed from ith vertex to jth vertex}, \\ 0, & \textit{otherwise}. \end{cases}$$

Example Consider the digraph D of Figure 11.26.

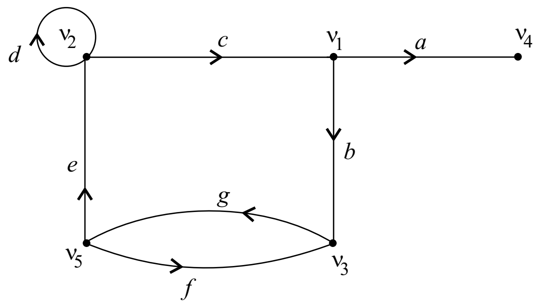


Fig. 11.26

The adjacency matrix of D is

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

We have the following observations about the adjacency matrix X of a digraph D.

1. X is a symmetric matrix if and only if D is a symmetric digraph.

2. Every non-zero element on the main diagonal element represents a loop at the corresponding vertex.

3. The parallel arcs cannot be represented by X and therefore X is defined only for a digraph without parallel arcs.

4. The sum of each row equals the outdegree of the corresponding vertex and the sum of each column equals the indegree of the corresponding vertex. The number of non-zero entries of X equals the number of arcs in D.

5. Permutation of any rows together with a permutation of the corresponding columns does not alter the digraph and thus the permutation corresponds to a reordering of the vertices. Therefore two digraphs are isomorphic if and only if their adjacency matrices differ only by such permutations.

6. If X is the adjacency matrix of a digraph D, then the transposed matrix X^T is the adjacency matrix of a digraph D^* obtained by reversing the direction of every arc in D.

7. For any $(0-1)$-matrix Q of order n, there exists a unique digraph D of n vertices such that Q is the adjacency matrix of D.

Connectedness and adjacency matrix: A digraph is disconnected if and only if its vertices can be ordered in such a way that its adjacency matrix X can be expressed as the direct sum of two square submatrices X_1 and X_2 as

$$X = \begin{bmatrix} X_1 & : & O \\ .. & : & .. \\ O & : & X_2 \end{bmatrix}. \tag{11.7.iii}$$

This partitioning is possible if and only if the vertices in the submatrix X_1 have no arc going to or coming from the vertex of X_2.

Similarly, a digraph is weakly connected if and only if its vertices can be ordered in such a way that its adjacency matrix can be expressed as

$$X = \begin{bmatrix} X_1 & : & O \\ .. & : & .. \\ X_{21} & : & X_2 \end{bmatrix} \tag{11.7.iv}$$

or $$X = \begin{bmatrix} X_1 & : & X_{12} \\ .. & : & .. \\ O & : & X_2 \end{bmatrix}, \tag{11.7.v}$$

where X_1 and X_2 are square submatrices.

Form (11.7.iv) represents the case when there is no arc going from the subdigraph corresponding to X_1 to the one corresponding to X_2. Form (11.7.v) represents the case when there is no arc going from the subdigraph corresponding to X_2 to the subdigraph corresponding to X_1.

Since a strongly connected digraph is neither disconnected nor weakly connected, a digraph is strongly connected if and only if the vertices of D cannot be ordered such that its adjacency matrix X is expressible in the form (11.7.iii), or (11.7.iv), or (11.7.v).

Theorem 11.22 $[X^k]_{ij}$ is the number of different arc sequences of k arcs from the ith vertex to the jth vertex.

Proof Induct on k. The result is trivially true for $k = 1$. Assume the result holds for $[X^{k-1}]_{ij}$. Now,

$$[X^k]_{ij} = [X^{k-1}X]_{ij} = \sum_{r=1}^{n} [X^{k-1}]_{ir}[X]_{rj} = \sum_{r=1}^{n} [X^{k-1}]_{ir}x_{r_j} \tag{11.22.1}$$

$$= \sum_{r=1}^{n} (\text{number of all directed arc sequences of length } k-1 \text{ from}$$

$$\text{vertex } i \text{ to } r\ )\ x_{rj},$$

by induction hypothesis.

In (11.22.1), $x_{rj} = 1$ or 0, according as there is an arc from r to j. Therefore a term in the sum (11.22.1) is non zero if and only if there is an arc sequence of length k from i to j, whose last arc is from r to j. If the term is non-zero, its value equals the number of such arc sequences from i to j through r. This holds for every vertex r, $1 \le r \le n$. Thus (11.22.1) is equal to the number of all possible arc sequences from i to j. ❑

It is to be noted that $[X^k]_{ij}$ gives the number of all arc sequences from vertex i to j and these arc sequences can be of the following types.

1. Directed paths from i to j, that is, those arc sequences in which no vertex is traversed more than once.
2. Directed walks from i to j, that is, those directed arc sequences in which a vertex may be traversed more than once, but no arc is traversed more than once.
3. Those arc sequences in which an arc may also be traversed more than once.

11.8 Number of Arborescences

We now give a formula for counting the number of spanning arborescence in a labeled, connected digraph (which of course is simple). First, we have the following definition.

Kirchoff matrix: For a digraph (simple) D of n vertices, the Kirchoff matrix is an $n \times n$ matrix $K(D) = [k_{ij}]$ defined by

$$k_{ij} = \begin{cases} d^-(v_i)\,, & i = j, \quad \text{in degree } of\ the\ ith\ vertex, \\ -x_{ij}\,, & i \neq j \quad (i,\ j)th\ entry\ in\ the\ adjacency\ matrix\,,\ with\ a\ negative\ sign\,. \end{cases}$$

Example Consider the digraph D given in Figure 11.27.

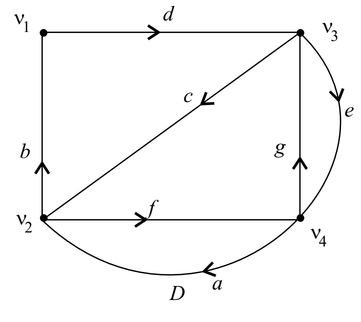


Fig. 11.27

The Kirchoff matrix of D is

$$K(D) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 1 & 0 & -1 & 0 \\ v_2 & -1 & 2 & 0 & -1 \\ v_3 & 0 & -1 & 2 & -1 \\ v_4 & 0 & -1 & -1 & 2 \end{array}$$

Clearly, the sum of the entries in each column in K is equal to zero, so that the n rows are linearly independent. Thus, $\det K = 0$.

Theorem 11.23 A digraph (simple) D of n vertices and $n-1$ arcs is an arborescence rooted at v_1 if and only if the (1, 1) cofactor of $K(D)$ is equal to 1.

Proof

a. Let D be an arborescence with n vertices and rooted at vertex v_1. Relabel the vertices as $v_1, v_2, \ldots, v_n$ such that vertices along every path from the root v_1 have increasing indices. Permute the rows and columns of $K(D)$ to conform with this relabelling.

   Since the indegree of v_1 equals zero, the first column contains only zeros. Other entries in $K(D)$ are

$$k_{ij} = \begin{cases} 0, & i > j, \\ -x_{ij}, & i < j, \\ 1, & i = j, \ i > 1. \end{cases}$$

   Then the K matrix of an arborescence rooted at v_1 is of the form

$$K(D) = \begin{bmatrix} 0 & -x_{12} & -x_{13} & -x_{14} & \cdot\cdot & -x_{1n} \\ 0 & 1 & -x_{23} & -x_{24} & \cdot\cdot & -x_{2n} \\ 0 & 0 & 1 & -x_{34} & \cdot\cdot & -x_{3n} \\ 0 & 0 & 0 & 1 & \cdot\cdot & \\ \vdots & & & & \cdot\cdot & \\ 0 & 0 & 0 & 0 & \cdot\cdot & 1 \end{bmatrix}.$$

   Clearly, the cofactor of the (1, 1) entry is 1, that is, $\det K_{11} = 1$.

b. Conversely, let D be a digraph of n vertices and $n-1$ arcs, and let (1, 1) cofactor of its K matrix be equal to 1, that is, $\det K_{11} = 1$.

   Since $\det K_{11} \neq 0$, every column in K_{11} has at least one non-zero entry. Therefore,

   $d^-(v_i) \geq 1$, for $i = 2, 3, \ldots, n$.

   There are only $n-1$ arcs to go around, therefore

$$d^-(v_i) = 1, \text{ for } i = 2, 3, \ldots, n, \text{ and } d^-(v_1) = 0.$$

Since no vertex in D has an indegree of more than one, if D can have any cycle at all, it has to be a directed cycle. Suppose that such a directed cycle exists, which passes through vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$. Then the sum of the columns $i_1, i_2, \ldots, i_r$ in K_{11} is zero. This is because each of these columns contains exactly two non-zero entries, as 1 on the main diagonal and -1 for the incoming arc from the vertex preceding it in the directed cycle. Thus the r columns in K_{11} are linearly dependent. So, det $K_{11} = 0$, a contradiction. Therefore, D has no cycles.

If D has $n-1$ arcs and no cycles, it must be a tree. Since in this tree $d^-(v_1) = 0$ and $d^-(v_i) = 1$, for $i = 2, 3, \ldots, n$, D is an arborescence rooted at vertex v_1.

The arguments in (a) and (b) are valid for an arborescence rooted at any vertex v_q. Any reordering of the vertices in D corresponds to identical permutations of rows and columns in $K(D)$. Such permutations do not alter the value or sign of the determinant. ❑

Theorem 11.24 If $K(D)$ is the Kirchoff matrix of a (simple) digraph D, then the value of the (q, q) cofactor of $K(D)$ is equal to the number of arborescences in D rooted at the vertex v_q.

Proof The proof depends on the result of Theorem 11.23 and on the fact that the determinant of a square matrix is a linear function of its columns. In particular, if P is a square matrix consisting of n column vectors, each of dimension n, that is

$$P = [p_1, p_2, \ldots, (p_i + p_i'), \ldots, p_n],$$

$$\text{then } \det P = \det\,[p_1, p_2, \ldots, p_i, \ldots, p_n] + \det\,[p_1, p_2, \ldots, p_i', \ldots, p_n]. \qquad (11.24.1)$$

In digraph D, suppose that vertex v_j has indegree d_j. The jth column of $K(D)$ can be regarded as the sum of d_j different columns, each corresponding to a digraph in which v_j has indegree one. And then (11.24.1) can be repeatedly applied. After this, splitting of columns can be carried out for each j, $j \neq q$ and det $K_{qq}(D)$ can be expressed as a sum of determinants of subdigraphs, that is

$$\det K_{qq}(D) = \sum_{D'} \det K_{qq}(D'), \qquad (11.24.2)$$

where D' is a subdigraph of D with the following properties.

1. Every vertex in D' has an indegree of exactly one except v_q.
2. D' has $n-1$ vertices and hence $n-1$ arcs.

From Theorem 11.23,

$$\det K_{qq}(D') = \begin{cases} 1, & \textit{if and only if } D' \textit{ is an arborescence rooted at } q\,, \\ 0, & \textit{otherwise}\,. \end{cases}$$

Thus the summation in (11.24.2) carried over all D''s equals the number of arborescences rooted at v_q. ❑

11.9 Tournaments

A tournament is an orientation of a complete graph. Therefore in a tournament each pair of distinct vertices v_i and v_j is joined by one and only one of the oriented arcs (v_i, v_j) or (v_j, v_i). If the arc (v_i, v_j) is in T, then we say v_i dominates v_j and is denoted by $v_i \to v_j$. The relation of dominance thus defined is a complete, irreflexive, antisymmetric binary relation. Figure 11.28 displays all tournaments on three and four vertices.

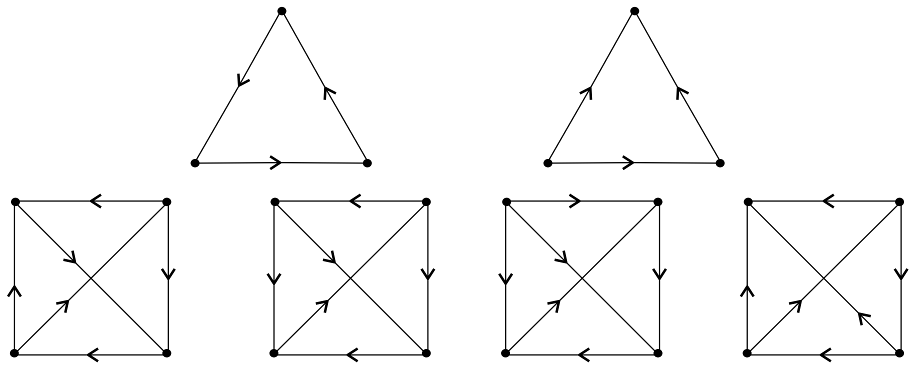


Fig. 11.28

Definition: A *triple* in a tournament T is the subdigraph induced by any three vertices. A triple (u, v, w) in T is said to be *transitive* if whenever $(u, v) \in A(T)$ and $(v, w) \in A(T)$, then $(u, w) \in A(T)$. That is, whenever $u \to v$ and $v \to w$, then $u \to w$.

Definition: A *bipartite tournament* is an orientation of a complete bipartite graph. A *k-partite tournament* is an orientation of a complete k-partite graph. Figure 11.29 displays a bipartite and a tripartite tournament.

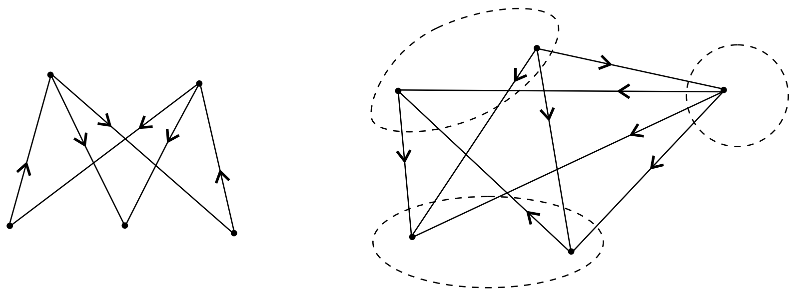


Fig. 11.29

Theorem 11.25 If v is a vertex having maximum outdegree in the tournament T, then for every vertex w of T there is a directed path from v to w of length at most 2.

Proof Let T be a tournament with n vertices and let v be a vertex of maximum outdegree in T. Let $d^+(v) = m$ and let $v_1, v_2, \ldots, v_m$ be the vertices in T such that there are arcs from v to v_i, $1 \le i \le m$. Since T is a tournament, there are arcs from the remaining $n-m-1$ vertices, say $u_1, u_2, \ldots, u_{n-m-1}$ to v. That is, there are arcs from u_j to v, $1 \le j \le n-m-1$ (Fig. 11.30).

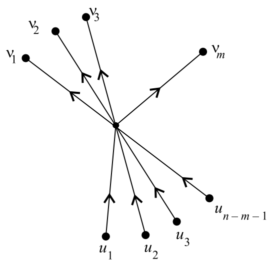


Fig. 11.30

Then for each i, $1 \le i \le m$, the arc from v to v_i gives a directed path of length 1 from v to v_i. We now show that there is a directed path of length 2 from v to u_j for each j, $1 \le j \le n-m-1$.

Given such a vertex u_j, if there is an arc from v_i to u_j for some i, then vv_iu_j is a directed path of length 2 from v to u_j. Now, let there be a vertex u_k, $1 \le k \le n-m-1$, such that no vertex $v_i, 1 \le i \le m$, has an arc from v_i to u_k. Since T is tournament, there is an arc from u_k to each of the m vertices v_i. Also, there is an arc from u_k to v and therefore $d^+(u_k) \ge m+1$. This contradicts the fact that v has maximum outdegree with $d^+(v) = m$. Thus each u_j must have an arc joining it from some v_i and the proof is complete by using the directed path vv_iu_j. ❑

Let T be a tournament with n vertices and let v be any vertex of T. Then $T-v$ is the digraph obtained from T by removing v and all arcs incident with v. Clearly, any two vertices of $T-v$ are joined by exactly one arc, since these two vertices are joined by exactly one arc in T. Thus $T-v$ is again a tournament.

Definition: A *directed Hamiltonian path* of a digraph D is the directed path in D that includes every vertex of D exactly once.

The following result, due to Redei [216], shows that a tournament contains a direct Hamiltonian path.

Theorem 11.26 (Redei) Every tournament T has a directed Hamiltonian path.

Proof Let T be a tournament with n vertices. We induct on n. When $n = 1$, 2, or 3, the result is trivially true (Fig. 11.31).

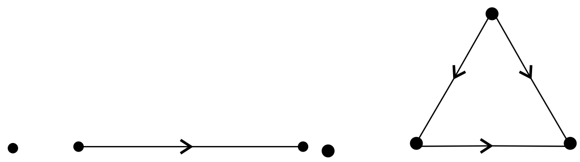


Fig. 11.31

Let $n \geq 4$. Assume that the result is true for all tournaments with fewer than n vertices. Let v be any vertex of T. Then $T - v$ is a tournament with $n - 1$ vertices and by induction hypothesis has a directed Hamiltonian path, say $P = v_1 v_2 \ldots v_{n-1}$.

In case there is an arc from v to v_1, then $P_1 = v v_1\ v_2 \ldots v_{n-1}$ is a directed Hamiltonian path in T. Similarly if there is an arc from v_{n-1} to v, then $P_2 = v_1\ v_2 \ldots v_{n-1} v$ is a directed Hamiltonian path in T.

Now, assume there is no arc from v to v_1 and no arc from v_{n-1} to v. Then there is at least one vertex w on the path P with the property that there is an arc from w to v and w is not v_{n-1}. Let v_i be the last vertex on P having this property, so that the next vertex v_{i+1} does not have this property. Then there is an arc from v_i to v and an arc from v to v_{i+1}, as shown in Figure 11.32. Thus $Q = v_1 v_2 \ldots v_i v\ v_{i+1}\ v_{i+2} \ldots v_{n-1}$ is a directed Hamiltonian path in T. ❑

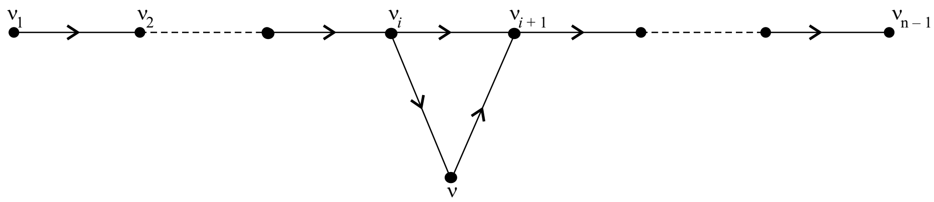


Fig. 11.32

Definition: A *directed Hamiltonian cycle* in a digraph D is a directed cycle which includes every vertex of D. If D contains such a cycle, then D is called *Hamiltonian*.

The next two results are due to Camion [44].

Theorem 11.27 (Camion) A strongly connected tournament T on n vertices contains cycles of length 3, 4,…, n.

Proof First we show that T contains a cycle of length three. Let v be any vertex of T. Let W denote the set of all vertices w of T for which there is an arc from v to w. Let Z denote the set of all vertices z of T for which there is an arc from z to v. We note that $W \bigcap Z = \varphi$, since T is a tournament.

Since T is strongly connected, W and Z are both nonempty. For, if W is empty, then there is no arc going out of v, which is impossible because T is strongly connected and the same argument can be used for Z. Again, because T is strongly connected, there is an arc in T going from some w in W to some z in Z. This gives the directed cycle $v\ w\ z\ v$ of length 3 (Fig. 11.33).

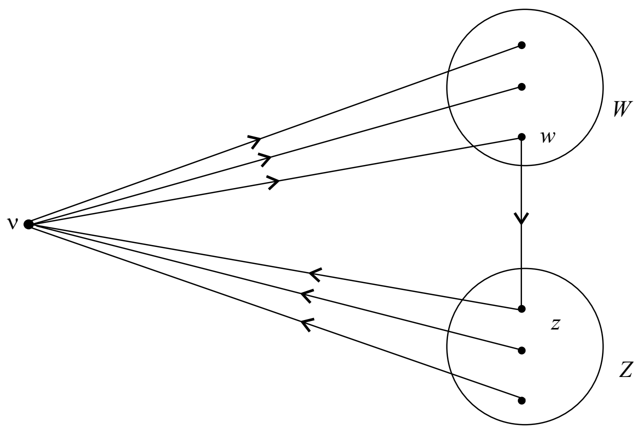


Fig. 11.33

Now induct on n. Assume T has a cycle C of length k, where $k < n$ and $k \geq 3$ and let this cycle be $v_1\, v_2\, \ldots\, v_k\, v_1$. We show that T has a cycle of length $k+1$.

Let there be a vertex v not on the cycle C, with the property that there is an arc from v to v_i and an arc from v_j to v for some v_i, v_j on C. Then there is a vertex v_i on C with an arc from v_{i-1} to v and an arc from v to v_i. Therefore, $C_1 = v_1 v_2 \ldots v_{i-1}\, v\, v_i v_{i+1} \ldots v_k v_1$ is a cycle of length $k+1$ (Fig. 11.34).

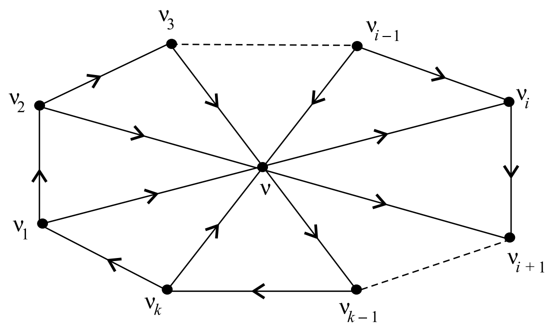


Fig. 11.34

If no vertex exists with the above property, then the set of vertices not contained in the cycle can be divided into two distinct sets W and Z, where W is the set of vertices w such that for each i, $1 \leq i \leq k$, there is an arc from v_i to w and Z is the set of vertices z such that for each i, $1 \leq i \leq k$, there is an arc from z to v_i. If W is empty then the vertices of C, and the vertices of Z together make up all the vertices in T. But, by definition of Z, there is no arc from a vertex on C to a vertex in Z, a contradiction, because T is strongly connected. Thus W is nonempty. A similar argument shows that Z is nonempty. Again, since T is strongly connected, there is an arc from some w in W to some z in Z. Then $C_1 = v_1\, w\, z\, v_3\, v_4 \ldots v_k\, v_1$ is a cycle of length $k+1$ (Fig. 11.35). This completes the proof. ❑

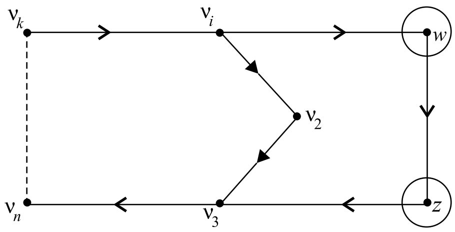


Fig. 11.35

Theorem 11.28 (Camion) A tournament T is Hamiltonian if and only if it is strongly connected.

Proof Let T have n vertices. If T is strongly connected, then by Theorem 11.27, T has a cycle of length n. Such a cycle is a Hamiltonian cycle, since it includes every vertex of T. Hence T is Hamiltonian.

Conversely, let T be Hamiltonian with Hamiltonian cycle $C = v_1 v_2 \ldots v_n v_1$. Then given any v_i, v_j with $i \geq j$, in the vertex set of T, $v_j v_{j+1} \ldots v_i$ is a path P_1 from v_j to v_i while $v_i v_{i+1} \ldots v_{n-1} v_n v_1 \ldots v_{j-1} v_j$ is a path P_2 from v_i to v_j (Fig. 11.36). Thus each vertex is reachable from any other vertex and so T is strongly connected. ❑

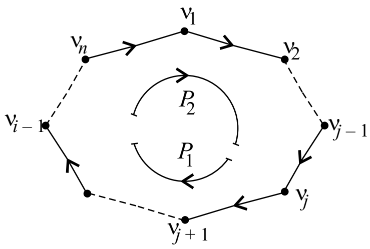


Fig. 11.36

11.10 Exercises

1. Prove that the converse of a strong digraph is also strong.

2. Show that D^*, the condensation of any digraph D, is cyclic.

3. Prove that the converse of a unilateral digraph is unilateral.

4. Show that the transmitters, receivers and isolates of a digraph D retain their properties in D^*.

5. Prove that the only acyclic digraph is $K_1 \cong K_1^*$.

6. Prove that an acyclic digraph without isolates has a transmitter and a receiver.

7. Show that every vertex v of a non-trivial digraph D has total even degree if and only if D is the union of arc-disjoint cycles.

8. Prove that every arc in a digraph belongs either to a directed cycle or a directed cut-set.

9. Prove that the digraph $D = (V, A)$ with $d_v^+ > 0$ for all $v \in V$, has a cycle.

10. Prove that the digraph distance satisfies the triangle inequality.

11. Prove that every Eulerian digraph is strong. Is the converse true?

12. If $E|G|$ is the number of Euler lines in an n-vertex Euler digraph D, show that $2^{n-1}.E|G|$ is the number of Euler lines in $L(D)$.

13. If D is a digraph with an odd number of vertices and if each vertex of D has an odd outdegree, prove that there is an odd number of vertices of D with odd indegree.

14. Prove by induction on n that for each $n \geq 1$, there is a simple digraph D with n vertices $v_1, v_2, \ldots, v_n$ such that $d_{v_i}^+ = i-1$ and $d_{v_i}^- = n-i$ for each $i = 1, 2, \ldots, n$.

15. Prove that no strictly weak digraph contains a vertex whose removal results in a strong digraph.

16. There exists a digraph with outdegree sequence $[s_1, s_2, \ldots, s_n]$, where $n-1 \geq s_1 \geq s_2 \geq \ldots \geq s_n$ and indegree sequences $[t_1, t_2, \ldots, t_n]$ where every $t_j \leq n-1$ if and only if $\sum s_i = \sum t_i$ and for each integer $k < n$,
$\sum\limits_{i=1}^{k} s_i \leq \sum\limits_{i=1}^{k} \min\{k-1,\ t_i\} + \sum\limits_{i=k+1}^{n} \min\{k,\ t_i\}$.

17. If X is the adjacency matrix of the edge digraph of a complete symmetric digraph, then $X^2 + X$ has all entries 1.

18. Let T be any tournament. Prove that the converse of T and the complement of T are isomorphic.

19. Prove that a tournament is transitive if and only if it has a unique Hamiltonian path.

20. Prove that if a simple digraph D has a cycle of length three, then it is not transitive.

21. Prove that a tournament is transitive if and only if it has no directed cycles.