USEFUL FORMULAS

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta
\]

\[
\sin \theta = S / D , \quad \cos \theta = C / D , \quad D > 0 \Rightarrow \theta = \arctan_2(S, C)
\]

\[
\partial (e^{\hat{\theta} t}) / \partial \theta = \hat{n} e^{\hat{\theta} t} \hat{n}
\]

\[
\hat{m} = e^{\hat{\beta} t} \hat{n} \Rightarrow \hat{m} = e^{\hat{\beta} t} \hat{n} e^{-\hat{\beta} t} \hat{n}
\]

\[
e^{\hat{\beta} t} \hat{u}_i = \hat{u}_i \quad \text{and} \quad e^{\hat{\beta} t} \hat{u}_i = \hat{u}_i^t
\]

\[
e^{\hat{\beta} t} \hat{u}_i = \hat{u}_j \cos \theta + \hat{n}_{ij} \sin \theta ; \quad \hat{n}_{ij} = \hat{u}_j \times \hat{u}_i, \quad i \neq j.
\]

\[
e^{\hat{\beta} t} \hat{u}_j = \hat{u}_j \cos \theta + \hat{n}_{ji} \sin \theta ; \quad \hat{n}_{ji} = \hat{u}_j \times \hat{u}_i, \quad i \neq j.
\]

\[
\hat{\omega}^{(a)}_{b/a} = \hat{C}^{(a,b)} \hat{C}^{(b,a)} \quad \text{and} \quad \hat{\omega}^{(b)}_{a/b} = \hat{C}^{(b,a)} \hat{C}^{(a,b)}
\]

\[
\hat{V}_p = \hat{J}_p \hat{q} \quad \text{where} \quad \hat{V}_p = \begin{bmatrix} \hat{V}_p \\ \hat{\omega} \end{bmatrix}
\]

\[
\hat{Q} = \hat{G} - \hat{J}_p \hat{p} \hat{P} \quad \text{where} \quad \hat{P} = \begin{bmatrix} \hat{F} \\ \hat{M}_p \end{bmatrix} \quad \text{and} \quad \hat{G} = \begin{bmatrix} \frac{\partial U}{\partial q_1} & \frac{\partial U}{\partial q_2} & \cdots \frac{\partial U}{\partial q_m} \end{bmatrix}
\]

\[
\hat{v}_p = \sum_{k=1}^{m} \hat{J}_p \hat{q}_k \quad \text{and} \quad \hat{\omega} = \sum_{k=1}^{m} \hat{J}_k \hat{q}_k
\]

\[
\hat{J}_p \hat{q} = \partial \hat{p} / \partial \hat{q}_k \quad \text{and} \quad \hat{J}_k = \text{column}[\hat{C} \hat{C} / \partial \hat{q}_k \hat{C}^t]
\]

\[
\hat{J}_p = \begin{bmatrix} \hat{J}_p & \hat{J}_p & \cdots & \hat{J}_p \\ \hat{J}_k & \hat{J}_k & \cdots & \hat{J}_k \end{bmatrix}
\]
Figure 1. An RPRRPR Manipulator

**PROBLEM 1**

Consider the RPRRPR manipulator shown in Figure 1 with its complete side view and a partial front view. The orientation of the hand and the location of the common tip and wrist points have already been expressed as follows with respect to the base frame.

\[
\hat{\mathbf{C}} = e^{\hat{\mathbf{u}}_1 \hat{\theta}_1} e^{\hat{\mathbf{u}}_2 \hat{\theta}_2} e^{\hat{\mathbf{u}}_3 \hat{\theta}_3} e^{-\hat{\mathbf{u}}_2 \hat{\theta}_3} e^{\hat{\mathbf{u}}_1 \pi/2} \\
\hat{\mathbf{p}} = e^{\hat{\mathbf{u}}_1 \hat{\theta}_1} \left[ \overline{\mathbf{u}}_1 (a_1 + d_4 s \theta_3 + s_s c \theta_3 \theta_4) - \overline{\mathbf{u}}_2 (s_s c \theta_4) + \overline{\mathbf{u}}_3 (s_2 + d_4 c \theta_3 - s_s s \theta_3 \theta_4) \right]
\]

The angular velocity of the hand and the velocity of the tip point can be expressed with respect to the base frame as written below.

\[
\hat{\omega} = \overline{\mathbf{\Omega}}_1 \hat{\theta}_1 + \overline{\mathbf{\Omega}}_2 \hat{\theta}_2 + \overline{\mathbf{\Omega}}_4 \hat{\theta}_4 + \overline{\mathbf{\Omega}}_6 \hat{\theta}_6 \\
\hat{\mathbf{v}} = e^{\hat{\mathbf{u}}_3 \hat{\theta}_3} (\overline{\mathbf{l}}_1 \hat{\theta}_1 + \overline{\mathbf{l}}_2 \hat{\theta}_2 + \overline{\mathbf{l}}_3 \hat{\theta}_3 + \overline{\mathbf{l}}_4 \hat{\theta}_4 + \overline{\mathbf{l}}_5 \hat{\theta}_5)
\]

Show that

\[
\overline{\mathbf{\Omega}}_1 = \overline{\mathbf{u}}_3 \, , \, \overline{\mathbf{\Omega}}_3 = e^{\hat{\mathbf{u}}_1 \hat{\theta}_1} \overline{\mathbf{u}}_2 \, , \, \overline{\mathbf{\Omega}}_4 = e^{\hat{\mathbf{u}}_1 \hat{\theta}_1} e^{\hat{\mathbf{u}}_2 \hat{\theta}_2} e^{\hat{\mathbf{u}}_3 \hat{\theta}_3} \, , \, \overline{\mathbf{\Omega}}_6 = -e^{\hat{\mathbf{u}}_1 \hat{\theta}_1} e^{\hat{\mathbf{u}}_2 \hat{\theta}_2} e^{\hat{\mathbf{u}}_3 \hat{\theta}_3} \overline{\mathbf{u}}_2 \\
\overline{\mathbf{l}}_1 = \overline{\mathbf{u}}_1 (s_s c \theta_4) + \overline{\mathbf{u}}_2 (a_1 + d_4 s \theta_3 + s_s c \theta_3 \theta_4) \\
\overline{\mathbf{l}}_2 = \overline{\mathbf{u}}_3 \\
\overline{\mathbf{l}}_3 = \overline{\mathbf{u}}_1 (d_4 c \theta_3 - s_s s \theta_3 \theta_4) - \overline{\mathbf{u}}_3 (d_4 s \theta_3 + s_s c \theta_3 \theta_4) \\
\overline{\mathbf{l}}_4 = \overline{\mathbf{u}}_1 (s_s c \theta_3 c \theta_4) + \overline{\mathbf{u}}_2 (s_s s \theta_4) - \overline{\mathbf{u}}_3 (s_s s \theta_3 c \theta_4) \\
\overline{\mathbf{l}}_5 = \overline{\mathbf{u}}_1 (c \theta_3 s \theta_4) - \overline{\mathbf{u}}_2 (c \theta_4) - \overline{\mathbf{u}}_3 (s \theta_3 s \theta_4)
SOLUTION

The given expression of $\hat{C}$ directly implies that

$$\vec{\omega} = \dot{\theta}_1 \vec{u}_3 + \dot{\theta}_2 e^{\ddot{\theta}_1} \vec{u}_2 + \dot{\theta}_4 e^{\ddot{\theta}_1} e^{\ddot{\theta}_2} \vec{u}_3 - \dot{\theta}_6 e^{\ddot{\theta}_1} e^{\ddot{\theta}_2} e^{\ddot{\theta}_3} \vec{u}_2$$

Hence, the coefficients are identified as $\hat{\Omega}_k$ for $k = 1, 3, 4, 6$.

The tip point velocity is obtained as described below.

$$\vec{v} = \sum_{k=1}^{5} (\partial \vec{p}/\partial q_k) \dot{q}_k = \sum_{k=1}^{5} \vec{H}_k \dot{q}_k \Rightarrow \vec{H}_k = \partial \vec{p}/\partial q_k$$

According to the given expression,

$$\vec{H}_1 = \partial \vec{p}/\partial \theta_1 = e^{\ddot{\theta}_1} \vec{u}_3 [\vec{u}_1 (\cdot \cdot \cdot) - \vec{u}_2 (\cdot \cdot \cdot) + \vec{u}_3 (\cdot \cdot \cdot)]$$

$$\vec{H}_1 = e^{\ddot{\theta}_1} [\vec{u}_2 (\cdot \cdot \cdot) + \vec{u}_3 (\cdot \cdot \cdot)] = e^{\ddot{\theta}_1} \vec{L}_1$$

$$\vec{L}_1 = \vec{u}_1 (s_2 c\theta_4) + \vec{u}_2 (a_1 + d_4 s\theta_3 + s_2 c\theta_3 s\theta_4)$$

$$\vec{H}_2 = \partial \vec{p}/\partial s_2 = e^{\ddot{\theta}_1} \vec{c}[\vec{u}_1 (\cdot \cdot \cdot) - \vec{u}_2 (\cdot \cdot \cdot) + \vec{u}_3 (\cdot \cdot \cdot)]/\partial s_2$$

$$\vec{H}_2 = e^{\ddot{\theta}_1} [\vec{L}_3]$$

$$\vec{L}_2 = \vec{u}_3$$

$$\vec{H}_3 = \partial \vec{p}/\partial \theta_3 = e^{\ddot{\theta}_3} \vec{u}_3 [\vec{u}_1 (d_4 c\theta_3 - s_5 s\theta_3 s\theta_4) - \vec{u}_3 (d_4 s\theta_3 + s_5 c\theta_3 s\theta_4)]$$

$$\vec{L}_3 = \vec{u}_1 (d_4 c\theta_3 - s_5 s\theta_3 s\theta_4) - \vec{u}_3 (d_4 s\theta_3 + s_5 c\theta_3 s\theta_4)$$

$$\vec{H}_4 = \partial \vec{p}/\partial \theta_4 = e^{\ddot{\theta}_4} \vec{c}[\vec{u}_1 (\cdot \cdot \cdot) - \vec{u}_2 (\cdot \cdot \cdot) + \vec{u}_3 (\cdot \cdot \cdot)]/\partial \theta_4$$

$$\vec{H}_4 = e^{\ddot{\theta}_4} [\vec{u}_1 (s_5 c\theta_3 c\theta_4) + \vec{u}_2 (s_2 s\theta_4) - \vec{u}_3 (s_5 s\theta_3 c\theta_4)]$$

$$\vec{L}_4 = \vec{u}_1 (s_5 c\theta_3 c\theta_4) + \vec{u}_2 (s_2 s\theta_4) - \vec{u}_3 (s_5 s\theta_3 c\theta_4)$$

$$\vec{H}_5 = \partial \vec{p}/\partial s_5 = e^{\ddot{\theta}_5} \vec{c}[\vec{u}_1 (\cdot \cdot \cdot) - \vec{u}_2 (\cdot \cdot \cdot) + \vec{u}_3 (\cdot \cdot \cdot)]/\partial s_5$$

$$\vec{H}_5 = e^{\ddot{\theta}_5} [\vec{u}_1 (c\theta_3 s\theta_4) - \vec{u}_2 (c\theta_4) - \vec{u}_3 (s_3 s\theta_4)]$$

$$\vec{L}_5 = \vec{u}_1 (c\theta_3 s\theta_4) - \vec{u}_2 (c\theta_4) - \vec{u}_3 (s_3 s\theta_4)$$
**PROBLEM 2**

Consider a special configuration of the manipulator in which all the angular joint variables except $\theta_3$ are zero. However, the rates of the joint variables are not necessarily zero. In this special configuration, show that

$$\dot{\omega} = \vec{u}_1(\dot{\theta}_4 \sin \theta_3) + \vec{u}_2(\dot{\theta}_3 - \dot{\theta}_6) + \vec{u}_3(\dot{\theta}_1 + \dot{\theta}_4 \cos \theta_3)$$

$$\vec{v} = \vec{u}_1[s_3 \dot{\theta}_1 + (d_4 \dot{\theta}_3 + s_5 \dot{\theta}_4) \cos \theta_3] + \vec{u}_2[(a_1 + d_4 s \theta_3) \dot{\theta}_1 - \dot{s}_5] + \vec{u}_3[s_2 - (d_4 \dot{\theta}_3 + s_5 \dot{\theta}_4) s \theta_3]$$

Then, determine the rates of the joint variables corresponding to specified $\dot{\omega}$ and $\vec{v}$.

Hint: Start from the $\dot{\omega}$ equation by treating $\dot{\theta}_3$ as if it is temporarily known. Then, proceed with the $\vec{v}$ equation.

As a by-product of the solution, determine the motion singularities, illustrate them by simple sketches, and discuss their consequences.

**SOLUTION**

The $\dot{\omega}$ equation leads to the following scalar equations.

$$\dot{\theta}_4 \sin \theta_3 = \omega_1, \quad \dot{\theta}_3 - \dot{\theta}_6 = \omega_2, \quad \dot{\theta}_1 + \dot{\theta}_4 \cos \theta_3 = \omega_3$$

These equations are solved as follows, if $\sin \theta_3 \neq 0$.

$$\dot{\theta}_6 = \dot{\theta}_3 - \omega_2, \quad \dot{\theta}_4 = \omega_1 / \sin \theta_3, \quad \dot{\theta}_1 = \omega_3 - \omega_1 / \tan \theta_3$$

If $\sin \theta_3 = 0$, the first kind of motion singularity occurs. With $\theta_3 = 0$ as the physically possible case, the scalar equations become

$$\omega_1 = 0, \quad \dot{\theta}_3 - \dot{\theta}_6 = \omega_2, \quad \dot{\theta}_1 + \dot{\theta}_4 = \omega_3$$

As noticed, this singularity necessitates that $\omega_1 = 0$ as a task space restriction, but brings extra freedom to the joint space by allowing arbitrary values for $\dot{\theta}_1$ and $\dot{\theta}_4$ provided that $\dot{\theta}_1 + \dot{\theta}_4 = \omega_3$. However, $\dot{\theta}_6$ is still obtained as $\dot{\theta}_6 = \dot{\theta}_3 - \omega_2$.

On the other hand, the $\vec{v}$ equation leads to the following scalar equations.

$$s_3 \dot{\theta}_1 + (d_4 \dot{\theta}_3 + s_5 \dot{\theta}_4) \cos \theta_3 = v_1$$

$$(a_1 + d_4 \sin \theta_3) \dot{\theta}_1 - \dot{s}_5 = v_2$$

$$\dot{s}_2 - (d_4 \dot{\theta}_3 + s_5 \dot{\theta}_4) \sin \theta_3 = v_3$$

If $\sin \theta_3 \neq 0$ and $\cos \theta_3 \neq 0$, these equations lead to the following solutions upon substituting the preceding solutions for $\dot{\theta}_1$ and $\dot{\theta}_4$.

$$\dot{\theta}_3 = \frac{v_1 - (s_5 \cos \theta_3)(\omega_1 / \sin \theta_3) - s_3(\omega_3 - \omega_1 / \tan \theta_3)}{d_4 \cos \theta_3} = \frac{v_1 - s_5 \omega_3}{d_4 \cos \theta_3}$$

$$\dot{s}_5 = (a_1 + d_4 \sin \theta_3)(\omega_3 - \omega_1 / \tan \theta_3) - v_2$$

$$\dot{s}_2 = v_3 + (d_4 \dot{\theta}_3 + s_5 \dot{\theta}_4) \sin \theta_3 = v_3 + (v_1 - s_5 \dot{\theta}_1) \tan \theta_3$$
If $\sin \theta_3 = 0$, with $\theta_3 = 0$, the preceding solution becomes modified into

$$\dot{\theta}_3 = (v_1 - s_5 \omega_3)/d_4$$
$$a_1 \dot{\theta}_4 - \dot{s}_5 = v_2$$
$$\dot{s}_2 = v_3$$

The second equation implies that, in addition to $\dot{\theta}_1$ and $\dot{\theta}_4$, $\dot{s}_5$ also becomes arbitrary within the constraint that $a_1 \dot{\theta}_1 - \dot{s}_5 = v_2$.

If $\cos \theta_3 = 0$, i.e. if $\theta_3 = \pm \pi/2$ or $\sin \theta_3 = \sigma'_3 = \pm 1$, the second kind of motion singularity occurs. In this singularity, the solutions to the $\omega$ equation become

$$\dot{\theta}_6 = \dot{\theta}_3 - \omega_2, \quad \dot{\theta}_4 = \sigma'_3 \omega_4, \quad \dot{\theta}_1 = \omega_5$$

Then, the scalar $\ddot{v}$ equations take the following forms, when the solutions to the $\omega$ equation are substituted.

$$s_5 \omega_3 = v_1$$
$$(a_1 + \sigma'_3 d_4) \omega_3 - \dot{s}_5 = v_2$$
$$\dot{s}_2 - \sigma'_3 d_4 \dot{\theta}_3 + \sigma'_3 s_5 (\sigma'_3 \omega_3) = \dot{s}_2 - \sigma'_3 d_4 \dot{\theta}_3 + s_5 \omega_1 = v_3$$

The second equation gives

$$\dot{s}_5 = (a_1 + \sigma'_3 d_4) \omega_3 - v_2$$

The first equation indicates the relationship between $v_1$ and $\omega_3$ as the task space restriction.

The $\omega_2$ and $v_3$ equations indicate the extra freedom arising in the joint space, which allows $\dot{\theta}_3$, $\dot{\theta}_6$, and $\dot{s}_2$ to be arbitrary within the constraints that

$$\dot{\theta}_3 - \dot{\theta}_6 = \omega_2 \quad \text{and} \quad \dot{s}_2 - \sigma'_3 d_4 \dot{\theta}_3 = v_3 - s_5 \omega_1$$

The singular configurations (physically realizable ones) are shown below.
PROBLEM 3

Obtain the Jacobian matrix of the manipulator in the special configuration described in Problem 2.

Write the transpose of the Jacobian matrix.

While the manipulator is in this configuration, the following force and moment are applied at the tip point.

\[
\vec{F} = F_1\vec{u}_1 + F_2\vec{u}_2 + F_3\vec{u}_3
\]

\[
\vec{M} = M_1\vec{u}_1 + M_2\vec{u}_2 + M_3\vec{u}_3
\]

These force and moment are so large that the weights of the links are negligible compared to them.

Find the actuator forces and torques \((T_1, f_2, T_3, T_4, f_5, T_6)\) in order to balance out \(\vec{F}\) and \(\vec{M}\).

SOLUTION

The \(\vec{v}\) and \(\vec{\omega}\) equations can be re-arranged as follows showing coefficients of the joint variable rates.

\[
\vec{v} = [u_1s_5 + u_2(a_1 + d_4s\theta_3)]\dot{\theta}_1 + u_3s_2 + (u_1c\theta_3 - u_3s\theta_3)d_4\dot{\theta}_3 + (u_1c\theta_3 - u_3s\theta_3)s_3\dot{\theta}_4 - u_2s_5
\]

\[
\vec{\omega} = u_3\dot{\theta}_1 + u_2\dot{\theta}_3 + (u_1\sin \theta_3 + u_3\cos \theta_3)\dot{\theta}_4 - u_2\dot{\theta}_6
\]

Hence, the Jacobian matrix is obtained as

\[
\begin{bmatrix}
 u_1s_5 + u_2(a_1 + d_4s\theta_3) & u_3 & (u_1c\theta_3 - u_3s\theta_3)d_4 & (u_1c\theta_3 - u_3s\theta_3)s_3 & -u_2 & 0 \\
 u_3 & u_2 & u_1\sin \theta_3 + u_3\cos \theta_3 & 0 & -u_2 \\
 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The transpose of the Jacobian matrix is

\[
\begin{bmatrix}
 u_1s_5 + u_2(a_1 + d_4s\theta_3) & u_3' \\
 u_3' & 0' \\
 (u_1c\theta_3 - u_3s\theta_3)d_4 & u_2' \\
 (u_1c\theta_3 - u_3s\theta_3)s_3 & u_3'\sin \theta_3 + u_3'\cos \theta_3 \\
 -u_2 & 0' \\
 0' & -u_2'
\end{bmatrix}
\]

The actuator forces and torques are found as follows.

\[
\vec{Q} = \vec{G} - J'\vec{P} \approx -J'\vec{P}
\]

\[
T_1 = -[s_5F_1 + (a_1 + d_4s\theta_3)F_2 + M_3], \quad f_2 = -F_3,
\]

\[
T_3 = -[(F_1c\theta_3 - F_3s\theta_3)d_4 + M_2],
\]

\[
T_4 = -[(F_1c\theta_3 - F_3s\theta_3)s_5 + M_1\sin \theta_3 + M_3\cos \theta_3]
\]

\[
f_5 = F_2, \quad T_6 = M_2.
\]