USEFUL FORMULAS

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta .
\]

\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta .
\]

\[
\sin \theta = S / D , \quad \cos \theta = C / D , \quad D > 0 \quad \Rightarrow \quad \theta = \tan^{-1}(S, C) .
\]

\[
(e \tilde{\theta})^{-1} = (e \tilde{\theta})^t = e^{-\tilde{\theta}} = e^{(-\tilde{\theta})} .
\]

\[
e^{\tilde{\theta}} \phi = e^{\tilde{\theta} + \tilde{\phi}} = e^{\tilde{\theta} + \tilde{\phi}} .
\]

\[
e^{\tilde{\theta}} \phi \neq e^{\tilde{\theta} + \tilde{\phi}} .
\]

\[
e^{\tilde{\theta}} \tilde{n} = \tilde{n} , \quad \tilde{t} = e^{\tilde{\theta}} = \tilde{t} ; \quad e^{\tilde{\theta}} \tilde{n} = \tilde{n} e^{\tilde{\theta}} .
\]

\[
e^{\tilde{\theta} + \pi / 2} = e^{\tilde{\theta} - \pi / 2} , \quad e^{\tilde{\theta} + \pi} = e^{\tilde{\theta}} , \quad e^{\tilde{\theta} + \pi} = e^{\tilde{\theta}} .
\]

\[
\bar{m} = e^{\tilde{\beta} \tilde{n}} \Rightarrow \bar{m} = e^{\tilde{\beta} \tilde{n}} e^{\tilde{\beta}} \quad \text{and} \quad e^{\tilde{\theta}} = e^{\tilde{\beta} \tilde{n}} e^{\tilde{\beta}} .
\]

\[
e^{\tilde{\theta}} \bar{u}_i = \bar{u}_i \quad \text{and} \quad \bar{u}_i = e^{\tilde{\theta}} .
\]

\[
e^{\tilde{\theta}} \bar{u}_j = \bar{u}_j \cos \theta + \bar{u}_j \sin \theta ; \quad \bar{u}_i = \bar{u}_j \bar{u}_j = \bar{u}_j \bar{u}_j , \quad i \neq j .
\]

\[
\bar{u}_j e^{\tilde{\theta}} = \bar{u}_j \cos \theta + \bar{u}_j \sin \theta ; \quad \bar{u}_i \bar{u}_i = \bar{u}_j \bar{u}_j , \quad i \neq j .
\]

\[
\omega_{a}^{(a)} = \hat{C}(a, b) \hat{C}(b, a) .
\]

\[
\bar{v}_p = \hat{J} \bar{q} \quad \text{and} \quad \bar{v}_R = \hat{J} \bar{q} \Rightarrow \bar{q} = \hat{J} \bar{p} \bar{p} \quad \text{and} \quad \bar{q} = \hat{J} \bar{R} \bar{R} .
\]
Consider the RPRPRR manipulator shown in the figure with its complete side view, partial top view, and partial inclined front view. The orientation of the hand and the location of the wrist point have already been expressed as follows with respect to the base frame.

\[ \hat{C} = e^{\tilde{\theta}_3 \theta_1} e^{\tilde{\theta}_4 \theta_2} e^{-\tilde{\theta}_5 \theta_6} \]

\[ \theta_3' = \theta_3 + \pi / 2 \]

\[ \varphi = e^{\tilde{\theta}_3 \theta_1} [\pi_2 (s_2 - s_4 s_3' - a_3 c_3') + \pi_3 (s_4 c_3' - a_3 s_3')] \]

\[ O = O_0 = O_1, \ E = O_2, \ Q = O_3, \ R = O_4 = O_5, \ P = O_6 \]
**PROBLEM 1 (20 points)**

Show that the velocity of the wrist point (\(\vec{w}\)) and the angular velocity of the hand (\(\vec{\omega}\)) can be obtained as follows from the preceding equations given for \(\vec{r}\) and \(\dot{\vec{C}}\).

\[
\vec{w} = e^\hat{\nu}_3 \hat{\nu}_1 (\vec{u}_l w_1 + \vec{u}_2 w_2 + \vec{u}_3 w_3)
\]

\[
w'_1 = (a_3 c \theta_3') + s_4 s \theta_3' - s_2) \hat{\theta}_1
\]

\[
w'_2 = \hat{s}_2 - s_4 s \theta_1' + (a_3 s \theta_3' - s_4 c \theta_3') \hat{\theta}_3
\]

\[
w'_3 = \hat{s}_4 c \theta_3' - (a_3 c \theta_3' + s_4 s \theta_3') \hat{\theta}_3
\]

\[
\vec{\omega} = \vec{u}_3 \hat{\theta}_1 + e^\hat{\nu}_3 \hat{\nu}_1 [\vec{u}_1 (\hat{\theta}_3 - \hat{\theta}_6 s \theta_3) - \vec{u}_2 \hat{\theta}_5 + \vec{u}_3 (\hat{\theta}_6 c \theta_3)]
\]

**SOLUTION**

\[
\vec{w} = \vec{r}
\]

\[
\vec{w} = \hat{\theta}_1 e^\hat{\nu}_3 \hat{\nu}_1 [\vec{u}_2 (s_2 - s_4 s \theta_1' - a_3 c \theta_3') + \vec{u}_3 (s_4 c \theta_3' - a_3 s \theta_3')]
\]

\[
\vec{w} = e^\hat{\nu}_3 \hat{\nu}_1 [\vec{u}_2 (s_2 - s_4 s \theta' + a_3 \theta_3' + s_4 \theta_3 s \theta_3') + \vec{u}_3 (s_4 c \theta_3' - a_3 \theta_3 c \theta_3')]
\]

\[
\vec{w} = e^\hat{\nu}_3 \hat{\nu}_1 [\vec{u}_1 w_1 + \vec{u}_2 w_2 + \vec{u}_3 w_3]
\]

\[
w'_1 = (a_3 c \theta_3') + s_4 s \theta_3' - s_2) \hat{\theta}_1
\]

\[
w'_2 = \hat{s}_2 - s_4 s \theta_1' + (a_3 s \theta_3' - s_4 c \theta_3') \hat{\theta}_3
\]

\[
w'_3 = \hat{s}_4 c \theta_3' - (a_3 c \theta_3' + s_4 s \theta_3') \hat{\theta}_3
\]

\[
\vec{\omega} = \text{col} [\hat{\vec{C}} \hat{\dot{\vec{C}}}^t]
\]

\[
\vec{\omega} = \hat{\theta}_1 \vec{u}_3 + \hat{\dot{\theta}}_2 e^{\hat{\nu}_3 \hat{\nu}_1} \vec{u}_1 - \hat{\dot{\theta}}_3 e^{\hat{\nu}_3 \hat{\nu}_1} e^{\hat{\nu}_1 \hat{\nu}_3} \vec{u}_2 + \hat{\dot{\theta}}_4 e^{\hat{\nu}_3 \hat{\nu}_1} e^{\hat{\nu}_1 \hat{\nu}_3} e^{-\hat{\nu}_2 \hat{\nu}_3} \vec{u}_3
\]

\[
\vec{\omega} = \hat{\theta}_1 \vec{u}_3 + \hat{\dot{\theta}}_2 e^{\hat{\nu}_3 \hat{\nu}_1} e^{\hat{\nu}_1 \hat{\nu}_3} \vec{u}_1 - \hat{\dot{\theta}}_3 e^{\hat{\nu}_3 \hat{\nu}_1} e^{\hat{\nu}_1 \hat{\nu}_3} \vec{u}_2 + \hat{\dot{\theta}}_4 e^{\hat{\nu}_3 \hat{\nu}_1} e^{\hat{\nu}_1 \hat{\nu}_3} e^{-\hat{\nu}_2 \hat{\nu}_3} \vec{u}_3
\]

\[
\vec{\omega} = \hat{\theta}_1 \vec{u}_3 + e^{\hat{\nu}_3 \hat{\nu}_1} e^{\hat{\nu}_1 \hat{\nu}_3} \vec{u}_1 (\hat{\theta}_3 - \hat{\theta}_6 s \theta_3) - \vec{u}_2 \hat{\theta}_5 + \vec{u}_3 (\hat{\theta}_6 c \theta_3)
\]

\[
\vec{\omega} = \vec{u}_3 \hat{\theta}_1 + e^{\hat{\nu}_3 \hat{\nu}_1} e^{\hat{\nu}_1 \hat{\nu}_3} [\vec{u}_1 (\hat{\theta}_3 - \hat{\theta}_6 s \theta_3) - \vec{u}_2 \hat{\theta}_5 + \vec{u}_3 (\hat{\theta}_6 c \theta_3)]
\]
**PROBLEM 2** (40 points)

a) Solve the equations given in Problem 1 for the rates of the joint variables.

b) Identify the motion singularities and discuss their consequences.

**SOLUTION**

a) Write the wrist velocity equation as

\[ \vec{u}_1 w_1' + \vec{u}_2 w_2' + \vec{u}_3 w_3' = e^{-\vec{u}_3 \dot{\theta}_1} \vec{w} \]

Premultiply both sides by \( \vec{u}_1^T \).

\[ w_1' = \vec{u}_1^T e^{-\vec{u}_3 \dot{\theta}_1} \vec{w} \rightarrow \quad \text{\( (a_3 c \theta_3' + s_4 s \theta_3') \dot{\theta}_1 = w_1 c \theta_1 + w_2 s \theta_1 \)} \]

If \( a_3 c \theta_3' + s_4 s \theta_3' - s_2 \neq 0 \), then

\[ \dot{\theta}_1 = \frac{w_1 c \theta_1 + w_2 s \theta_1}{a_3 c \theta_3' + s_4 s \theta_3' - s_2} \]

Having found \( \dot{\theta}_1 \), write the angular velocity as

\[ \vec{u}_1 (\dot{\theta}_3 - \dot{\theta}_6 s \theta_5) - \vec{u}_2 \dot{\theta}_5 + \vec{u}_3 (\dot{\theta}_6 c \theta_5) = \vec{\omega}' \]

where \( \vec{\omega}' \) is known as

\[ \vec{\omega}' = e^{-\vec{u}_1 \dot{\theta}_5} e^{-\vec{u}_3 \dot{\theta}_1} (\vec{\omega} - \vec{u}_2 \dot{\theta}_6 c \theta_5) \]

The previous equation leads to the following scalar equations.

\[ \dot{\theta}_3 - \dot{\theta}_6 \sin \theta_5 = \omega_1', \quad \dot{\theta}_5 = -\omega_2', \quad \dot{\theta}_6 \cos \theta_5 = \omega_5' \]

If \( \cos \theta_5 \neq 0 \), these equations are solved as follows.

\[ \dot{\theta}_5 = -\omega_2', \quad \dot{\theta}_6 = \omega_5' / \cos \theta_5, \quad \dot{\theta}_3 = \omega_1' + \dot{\theta}_6 \sin \theta_5 = \omega_1' + \omega_2' \tan \theta_5 \]

Having found these angles, the remaining two scalar equations for the wrist point velocity are written as

\[ w_2' = \vec{u}_2^T e^{-\vec{u}_3 \dot{\theta}_1} \vec{w} \rightarrow \quad s_2 - s_4 s \theta_3' + (a_3 s \theta_3' - s_4 c \theta_3') \dot{\theta}_3 = w_2 c \theta_1 - w_1 s \theta_1 \]

\[ w_3' = \vec{u}_3^T e^{-\vec{u}_3 \dot{\theta}_1} \vec{w} \rightarrow \quad s_4 c \theta_3' - (a_3 c \theta_3' + s_4 s \theta_3') \dot{\theta}_3 = w_3 \]

If \( \cos \theta_3' \neq 0 \), the last equation gives

\[ s_4 = \frac{w_3 + (a_3 c \theta_3' + s_4 s \theta_3') \dot{\theta}_3}{c \theta_3'} \]

Finally, the other equation gives

\[ s_2 = w_2 c \theta_1 - w_1 s \theta_1 + s_4 s \theta_3' - (a_3 s \theta_3' - s_4 c \theta_3') \dot{\theta}_3 \]
The solution obtained in Part (b), indicates three kinds of motion singularities. They are explained below.

1. The first kind of motion singularity occurs if \( a_3 c \theta_3^2 + s_4 s \theta_3^2 - s_2 = 0 \). In this singularity, \( \dot{\theta}_1 \) becomes arbitrary as an extra freedom in the joint space and wrist motion becomes constrained as \( w_1 c \theta_1 + w_2 s \theta_1 = 0 \) as a restriction in the task space.

2. The second kind of motion singularity occurs if \( \cos \theta_5 = 0 \), i.e. if \( \theta_5 = \pm \pi / 2 \). In this singularity, \( \dot{\theta}_3 \) and \( \dot{\theta}_6 \) become arbitrary as an extra freedom in the joint space and wrist motion becomes constrained as \( \omega_3 = 0 \) as a restriction in the task space. However, the sum or difference of \( \dot{\theta}_3 \) and \( \dot{\theta}_6 \) can still be obtained as \( \dot{\theta}_3 - \sigma_5 \dot{\theta}_6 = \omega_3^0 \) where \( \sigma_5 = \text{sgn} (\sin \theta_3) \).

3. The first kind of motion singularity occurs if \( \cos \theta_5^0 = 0 \), i.e. if \( \theta_5^0 = \pm \pi / 2 \). In this singularity, \( s_4 \) and \( s_2 \) become arbitrary as an extra freedom in the joint space and wrist motion becomes constrained as \( w_3 + (a_3 c \theta_3^2 + s_4 s \theta_3^2) \dot{\theta}_3 = 0 \) as a restriction in the task space. However, the sum or difference of \( \dot{\theta}_3 \) and \( \dot{\theta}_6 \) can still be obtained as \( \dot{\theta}_3 - \sigma_3 \dot{s}_4 = (w_2 c \theta_1 - w_1 s \theta_1) + \sigma_3^0 (a_3 \dot{\theta}_3) \) where \( \sigma_3^0 = \text{sgn} (\sin \theta_3) \).

**PROBLEM 3 (40 points)**

a) Recall that \( \vec{w} \) and \( \vec{\omega} \) can also be expressed as follows in terms of the velocity influence coefficients.

\[
\vec{w} = \sum_{k=1}^{6} J_{Rk} \dot{q}_k \quad \text{and} \quad \vec{\omega} = \sum_{k=1}^{6} J_{Ak} \dot{q}_k
\]

Regarding the equations given in Problem 1 from this point of view, identify the velocity influence coefficients.

b) Sketch the manipulator when \( \theta_1 = \theta_3 = 0 \) and \( \theta_3 = \pi / 2 \), i.e. \( \theta_3^0 = \pi \). Construct the wrist point Jacobian matrix (\( J_R \)) of the manipulator in this position.

c) The manipulator is carrying out a polishing task. In the position specified in Part (b), it presses down the polishing tool with a force \( H_3 \) and pushes it slowly in a lateral direction with the forces \( L_1 \) and \( L_2 \) respectively along the first and second axes of the base frame. At the same time, it spins the polishing tool about the vertical axis with a moment \( M_3 \). The polishing tool is attached to the hand by aligning its spin axis with the approach vector. The distance between the wrist point and tip point of the polishing tool is \( d_6 \). Determine the torques and forces to be applied by the joint actuators (which are \( T_1, F_2, T_3, F_4, T_5, T_6 \)) in order to produce the task forces and moments as described herein. You may ignore the effect of gravity.
SOLUTION

a) The equations given in Problem 1 imply the following velocity influence coefficients.

\[ \mathcal{J}_{R1} = e^{\hat{\theta}_3 \theta_1} \overline{u}_3 (a_3 c \theta_1^2 + s_4 s \theta_1^2 - s_2) \]
\[ \mathcal{J}_{R2} = e^{\hat{\theta}_1 \theta_1} \overline{u}_2 \]
\[ \mathcal{J}_{R3} = e^{\hat{\theta}_3 \theta_1} [ \overline{u}_2 (a_3 s \theta_2 + s_4 c \theta'_2) - \overline{u}_3 (a_3 c \theta_3 + s_4 s \theta'_3)] \]
\[ \mathcal{J}_{R4} = e^{\hat{\theta}_3 \theta_1} (\overline{u}_3 c \theta_3' - \overline{u}_2 s \theta_2') \]
\[ \mathcal{J}_{R5} = \mathcal{J}_{R6} = \overline{0} \]
\[ \mathcal{J}_{A1} = \overline{u}_3 \]
\[ \mathcal{J}_{A2} = \mathcal{J}_{A4} = \overline{0} \]
\[ \mathcal{J}_{A3} = e^{\hat{\theta}_3 \theta_1} \overline{u}_1 \]
\[ \mathcal{J}_{A5} = -e^{\hat{\theta}_3 \theta_1} e^{\hat{\theta}_1 \theta_1} \overline{u}_2 \]
\[ \mathcal{J}_{A6} = e^{\hat{\theta}_3 \theta_1} e^{\hat{\theta}_1 \theta_1} (\overline{u}_3 c \theta_5 - \overline{u}_1 s \theta_5) \]

b) In the required position, the velocity influence coefficients become

\[ \mathcal{J}_{R1} = -\overline{u}_1 (s_2 + a_3) \]
\[ \mathcal{J}_{R2} = \overline{u}_2 \]
\[ \mathcal{J}_{R3} = \overline{u}_2 s_4 + \overline{u}_3 a_3 \]
\[ \mathcal{J}_{R4} = -\overline{u}_3 \]
\[ \mathcal{J}_{R5} = \mathcal{J}_{R6} = \overline{0} \]
\[ \mathcal{J}_{A1} = \overline{u}_3 \]
\[ \mathcal{J}_{A2} = \mathcal{J}_{A4} = \overline{0} \]
\[ \mathcal{J}_{A3} = \overline{u}_1 \]
\[ \mathcal{J}_{A5} = \overline{u}_2 \]
\[ \mathcal{J}_{A6} = -\overline{u}_3 \]

Hence, the wrist point Jacobian matrix is constructed as

\[
\mathbf{\dot{J}}_R = \begin{bmatrix}
-(s_2 + a_3) & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & s_4 & 0 & 0 & 0 \\
0 & 0 & a_3 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]
c) For the described task, the force and moment vectors applied by the end-effector at and about its tip point are specified as follows with their base-frame components.

\[
F^{(0)} = \begin{bmatrix} L_1 \\ L_2 \\ -H_3 \end{bmatrix} \quad \text{and} \quad \overline{M}^{(0)}_p = \begin{bmatrix} 0 \\ 0 \\ M_3 \end{bmatrix}
\]

On the other hand, the Jacobian is available about the wrist point. So, the moment vector about the wrist point is obtained as follows.

\[
\overline{M}_R = \overline{M}_p + d_6 \bar{u}_d \times \bar{F} \quad \rightarrow \quad \overline{M}^{(0)}_R = \overline{M}^{(0)}_p + d_6 \bar{u}_d^{(0)} F^{(0)}
\]

In the specified position of the manipulator,

\[
\bar{u}_d^{(0)} = -\bar{w}_3
\]

Therefore,

\[
\overline{M}^{(0)}_R = \begin{bmatrix} 0 \\ 0 \\ -d_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ M_3 \end{bmatrix} - \begin{bmatrix} -L_2 \\ L_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ M_3 \end{bmatrix} = \begin{bmatrix} d_6 L_2 \\ -d_6 L_1 \\ -d_6 L_4 \\ M_3 \end{bmatrix}
\]

Hence, the augmented forcing vector at the wrist point becomes

\[
\overline{R} = \overline{R}^{(0)} = \begin{bmatrix} L_1 \\ L_2 \\ -H_3 \\ d_6 L_2 \\ -d_6 L_4 \\ M_3 \end{bmatrix}
\]

This forcing is created by the following actuator forces and moments.

\[
\overline{Q} = \bar{j}^T_R \overline{R}
\]

| \begin{bmatrix} T_1 \\ F_2 \\ T_3 \\ F_2 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} -(s_2 + a_3) & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & s_4 & a_3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ -H_3 \\ d_6 L_2 \\ -d_6 L_4 \\ M_3 \end{bmatrix} = \begin{bmatrix} M_3 - (s_2 + a_3) L_4 \\ L_2 \\ (s_4 + d_6) L_2 - a_3 H_3 \\ -H_3 \\ -d_6 L_4 \\ -M_3 \end{bmatrix} |