Interval oscillation of a general class of second-order nonlinear differential equations with nonlinear damping

A. Tiryaki\textsuperscript{a}, A. Zafer\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a}Department of Mathematics, Faculty of Arts and Sciences, Gazi University, 06500 Teknikokullar, Ankara, Turkey
\textsuperscript{b}Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey

Received 5 April 2004; accepted 12 August 2004

Abstract

The paper is concerned with the oscillation of a class of general type second order differential equations with nonlinear damping terms. Several new interval oscillation criteria are established for such a class of differential equations under quite general assumptions. Examples are also given to illustrate the results. In particular, it is shown that under some very mild conditions on \( k_1, k_2, \) and \( f \) the equation

\[(k_1(x, x'))' + \cos tk_2(x, x')x' + \sin tf(x) = 0\]


© 2004 Elsevier Ltd. All rights reserved.

Keywords: Nonlinear differential equation; Nonlinear damping; Oscillation

* Corresponding author. Tel.: +90-3122102995; fax: +90-3122101282.
E-mail address: zafer@metu.edu.tr (A. Zafer).
URL: http://www.math.metu.edu.tr/~zafer.
1. Introduction and preliminaries

We are concerned with oscillation of solutions of a class of second-order nonlinear differential equations with nonlinear damping of the form

\[(r(t)k_1(x, x'))' + p(t)k_2(x, x')x' + q(t)f(x) = 0, \quad t \geq t_0,\]

(1.1)

where \(t_0 \geq 0\) is a fixed real number, \(p, q : [t_0, \infty) \rightarrow \mathbb{R}, r : [t_0, \infty) \rightarrow (0, \infty), f : \mathbb{R} \rightarrow \mathbb{R},\) and \(k_1, k_2 : \mathbb{R}^2 \rightarrow \mathbb{R}.\) It is tacitly assumed that the functions \(p, q, f,\) and \(k_2\) are continuous, \(r\) and \(k_1\) are continuously differentiable in their domain of definitions. We restrict our attention to those solutions \(x(t)\) of (1.1) which exist on \([t_0, \infty)\) and satisfy \(\sup\{|x(t)| : t \geq t_0\} \neq 0\) for any \(t_0 \geq t_0.\) As usual, such a solution of Eq. (1.1) is called oscillatory if the set of its zeros is unbounded from above, otherwise it is said to be nonoscillatory. Eq. (1.1) is called oscillatory if all solutions are oscillatory.

The oscillation of Eq. (1.1) has been first studied by Rogovchenko and Rogovchenko [20]. Later, motivated by this paper the present authors have obtained several oscillation criteria for solutions of (1.1) in [23] under certain relationships between \(k_1\) and \(k_2.\) In both of these works, the conditions in terms of the coefficients involving integral averages over the whole half-line \([t_0, \infty)\) are used. More results in this direction can be found in [1,2,4–8,10,18–21,23,26–28] for the special cases of (1.1). As pointed out earlier [3,11], oscillation is an interval property, that is, it is more reasonable to investigate solutions on an infinite set of bounded intervals. The problem is therefore to find oscillation criteria which use only the information about the involved functions on these intervals, outside of these intervals the behavior of the functions is irrelevant. Such type of oscillation criteria are referred to as the interval oscillation criteria, see [3,9,11,12,14–17,25]. The first attempt in this direction is due to El-Sayed [3], who investigated linear second-order differential equations of the form

\[(r(t)x')' + q(t)x = g(t),\]

where \(q\) is of arbitrary sign and \(g\) is an oscillatory function, and gave an affirmative answer to an open problem of Wong stated in [24]. In 1999, Wong [25] substantially improved the results of El-Sayed with a more direct and simpler proof. Meanwhile, some interval oscillation criteria were also given by Huang [9] for the case \(g(t) \equiv 0, r(t) \equiv 1,\) and \(q(t) \geq 0\) for \(t \in [t_0, \infty).\) It should also be mentioned that Kwong and Zettl proved some partially related results much earlier in [12]. Recently, Agarwal and Li [15–17] and Sun [22] have presented several interval oscillation criteria for second-order nonlinear equations of the form

\[(r(t)x')' + p(t)x' + q(t)f(x) = 0,\]

where \(f'(x) \geq \mu_1 \) or \(xf(x) \geq \mu_2 x^2\) for some positive constants \(\mu_1\) and \(\mu_2\) and for all \(x \neq 0,\) and there is a nonnegativity condition imposed on \(q,\) especially when \(xf(x) \geq \mu_2 x^2.\) In our case, we also impose the same conditions on the function \(f\) appearing in (1.1) and derive interval oscillation criteria similar to ones obtained in [3,9,11,12,14–17,25]. The sign condition on \(q\) is dropped.
Since Eq. (1.1) includes as special cases quite a large class of equations in the literature, the results of this paper extend, improve, and generalize many oscillation criteria previously obtained. In particular we will see that under certain conditions the equation
\[(k_1(x, x')' + \cos tk_2(x, x')x' + \sin tf(x) = 0\]
is oscillatory. In the case \(k_1(u, v) = v, k_2(u, v) \equiv 1\), and \(f(x) = x\), this result was obtained by Kwong and Wong [13].

It is no doubt that the Riccati substitution and its generalized forms play a very important role in the oscillation theory of second-order equations. In this work together with the so-called \(H\)-method introduced by Philos [19] we shall also employ a generalized Riccati substitution of the form
\[w(t) = \rho(t) \left[ \frac{r(t)k_1(x(t), x'(t))}{F(x(t))} + g(t) \right], \quad F(x) = f(x) \text{ or } F(x) = x\]
to derive several interval oscillation criteria for (1.1). We will also see that the obtained oscillation criteria are applicable without any change in case there is a forcing term or rather a perturbation term on the right-hand side of (1.1).

Let us begin with a lemma which will substantially simplify the proofs of our results. First we recall a class functions defined on \(D = \{(t, s) : t \geq s \geq t_0\}\). A function \(H \in C(D, \mathbb{R})\) is said to belong to the class \(\mathcal{P}\) if

(i) \(H(t, t) = 0\) for \(t \geq t_0\) and \(H(t, s) > 0\) when \(t \neq s\);
(ii) \(H(t, s)\) has partial derivatives on \(D\) such that
\[
\frac{\partial H}{\partial t}(t, s) = h_1(t, s)\sqrt{H(t, s)}, \quad \frac{\partial H}{\partial s}(t, s) = -h_2(t, s)\sqrt{H(t, s)}
\]
for some \(h_1, h_2 \in L^1_{loc}(D, \mathbb{R})\).

**Lemma 1.1.** Let \(A_0, A_1, A_2 \in C([t_0, \infty), \mathbb{R})\) with \(A_2 > 0\), and \(w \in C^1([t_0, \infty), \mathbb{R})\). If there exist \((a, b) \subset [t_0, \infty)\) and \(c \in (a, b)\) such that
\[w' \leq -A_0(s) + A_1(s)w - A_2(s)w^2, \quad s \in (a, b), \quad (1.2)\]
then
\[
\frac{1}{H(c, a)} \int_a^c \left[ H(s, a)A_0(s) - \frac{1}{4A_2(s)} \Phi_1^2(s, a) \right] ds
+ \frac{1}{H(b, c)} \int_c^b \left[ H(b, s)A_0(s) - \frac{1}{4A_2(s)} \Phi_2^2(b, s) \right] ds \leq 0 \quad (1.3)
\]
for every \(H \in \mathcal{P}\), where
\[
\Phi_1(s, a) = h_1(s, a) + A_1(s)\sqrt{H(s, a)}; \quad \Phi_2(b, s) = h_2(b, s) - A_1(s)\sqrt{H(b, s)}.
\]
Proof. Multiplying (1.2) by \( H(s, t) \) and integrating with respect to \( s \) from \( t \) to \( c \) for \( t \in (a, c] \), we have
\[
\int_{t}^{c} H(s, t) A_0(s) \, ds \leq - \int_{t}^{c} H(s, t) w'(s) \, ds + \int_{t}^{c} H(s, t) A_1(s) w(s) \, ds \\
- \int_{c}^{t} H(s, t) A_2(s) w^2(s) \, ds.
\] (1.4)
In view of (i) and (ii), we see that
\[
\int_{t}^{c} H(s, t) w'(s) \, ds = H(c, t) w(c) - \int_{t}^{c} h_1(s, t) \sqrt{H(s, t)} w(s) \, ds.
\] (1.5)
Using (1.5) in (1.4) leads to
\[
\int_{t}^{c} H(s, t) A_0(s) \, ds \\
\leq - H(c, t) w(c) - \int_{t}^{c} \left[ A_2(s) H(s, t) w^2(s) - (h_1(s, t) \sqrt{H(s, t)} - A_1(s) H(s, t)) w(s) \right] \, ds \\
\quad + \int_{t}^{c} \frac{1}{4A_2(s)} \Phi_1^2(s, t) \, ds \\
\leq - H(t, c) w(c) + \int_{t}^{c} \frac{1}{4A_2(s)} \Phi_1^2(s, t) \, ds.
\] (1.6)
Similarly, if (1.2) is multiplied by \( H(t, s) \) and then integrated from \( c \) to \( t \) for \( t \in [c, b) \), then one gets
\[
\int_{c}^{t} H(t, s) A_0(s) \, ds \\
\leq H(t, c) w(c) - \int_{c}^{t} \left[ A_2(s) H(t, s) w^2(s) + (h_2(s, t) \sqrt{H(t, s)}) - A_1(s) H(t, s)) w(s) \right] \, ds \\
\quad + \int_{c}^{t} \frac{1}{4A_2(s)} \Phi_2^2(t, s) \, ds \\
\leq H(t, c) w(c) + \int_{c}^{t} \frac{1}{4A_2(s)} \Phi_2^2(t, s) \, ds.
\] (1.7)
Letting \( t \to a^+ \) in (1.6) and \( t \to b^- \) in (1.7) and adding the resulting inequalities we have (1.3). □
2. \( f(x) \) is differentiable

We shall assume that the following conditions are satisfied:

(C1) \( f(x) \) is differentiable, \( xf(x) \neq 0 \) and \( f'(x) \geq \mu_1 \) for some constant \( \mu_1 > 0 \) and all \( x \in \mathbb{R}\backslash\{0\} \);

(C2) \( k_1^2(u, v) \leq z_1 v k_1(u, v) \) for some \( z_1 > 0 \) and all \( (u, v) \in \mathbb{R}^2 \);

(C3) \( \nu f(u) k_2(u, v) \geq z_2 k_1^2(u, v) \) for some \( z_2 > 0 \) and all \( (u, v) \in \mathbb{R}^2 \).

**Theorem 2.1.** Let (C1), (C2), and (C3) hold. Suppose that there exists an interval \((a, b) \subset [t_0, \infty)\) such that \( p(t) \geq 0 \) for all \( t \in (a, b) \), and that there exist \( c \in (a, b), H \in \mathcal{P}, \rho \in C^1([t_0, \infty), (0, \infty)) \) and \( g \in C^1([t_0, \infty), \mathbb{R}) \) such that

\[
\frac{1}{H(c, a)} \int_a^c \left[ H(s, a) \rho(s) \phi(s) - \frac{z_1 \rho(s) r_2(s)}{4[z_1 z_2 p(s) + \mu_1 r(s)]} \Phi_1^2(s, a) \right] ds \\
+ \frac{1}{H(b, c)} \int_c^b \left[ H(b, s) \rho(s) \phi(s) - \frac{z_1 \rho(s) r_2(s)}{4[z_1 z_2 p(s) + \mu_1 r(s)]} \Phi_2^2(b, s) \right] ds > 0,
\]

(2.1)

where

\[
\phi(s) = q(s) + g^2(s) \frac{z_1 z_2 p(s) + \mu_2 r(s)}{z_1 r_2(s)} - g'(s),
\]

\[
\Phi_1(s, a) = h_1(s, a) + \left[ \frac{\rho'(s)}{\rho(s)} + 2 g(s) \frac{z_1 z_2 p(s) + \mu_2 r(s)}{z_1 r_2(s)} \right] \sqrt{H(s, a)},
\]

\[
\Phi_2(b, s) = h_2(b, s) - \left[ \frac{\rho'(s)}{\rho(s)} + 2 g(s) \frac{z_1 z_2 p(s) + \mu_2 r(s)}{z_1 r_2(s)} \right] \sqrt{H(b, s)}.
\]

(2.2)

Then every solution of (1.1) has a zero in \((a, b)\).

**Proof.** Otherwise, \( x(t) \neq 0 \) for all \( t \in (a, b) \). Define

\[
w(t) = \rho(t) \left[ \frac{r(t) k_1(x(t), x'(t))}{f(x(t))} + g(t) \right], \quad t \in (a, b).
\]

(2.3)

In view of (1.1), it follows from (2.3) that

\[
w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) \frac{k_2(x(t), x'(t)) x'(t)}{f(x(t))} \\
- \rho(t) q(t) - \rho(t) r(t) k_1(x(t), x'(t)) \frac{f'(x(t)) x'(t)}{f_2^2(x(t))} + \rho(t) g'(t).
\]

(2.4)
Applying Lemma 1.1 to (2.5) we see that inequality (2.1) fails to hold. □

Let Theorem 2.2.

Proof. Suppose that

\[ w'(t) \leq -\rho(t)\phi(t) + \left[ \frac{\rho'(t)}{\rho(t)} + 2g(t) \frac{\alpha_1\alpha_2 p(t) + \mu_1 r(t)}{\alpha_1 r^2(t)} \right] w(t) \]

\[ - \frac{\alpha_1\alpha_2 p(t) + \mu_1 r(t)}{\alpha_1 \rho(t) r^2(t)} w^2(t). \]

Comparing inequalities (1.2) and (2.5) we identify that

\[ A_0(t) = \rho(t)\phi(t); \]

\[ A_1(t) = \frac{\rho'(t)}{\rho(t)} + 2g(t) \frac{\alpha_1\alpha_2 p(t) + \mu_1 r(t)}{\alpha_1 r^2(t)}; \]

\[ A_2(t) = \frac{\alpha_1\alpha_2 p(t) + \mu_1 r(t)}{\alpha_1 \rho(t) r^2(t)}. \]

Applying Lemma 1.1 to (2.5) we see that inequality (2.1) fails to hold.

If the conditions of Theorem 2.1 hold for a sequence \( \{(a_n, b_n)\} \) of intervals such that \( \lim_{n \to \infty} a_n = \infty \), then we may conclude that (1.1) is oscillatory. That is, we have the following corollary.

**Corollary 2.1.** If for a given \( T \geq t_0 \) there exists an interval \( (a, b) \subset [T, \infty) \) for which the conditions of Theorem 2.1 are satisfied, then (1.1) is oscillatory.

The following theorem is also a consequence of Theorem 2.1.

**Theorem 2.2.** Let (C1), (C2), and (C3) hold, and \( p(t) \geq 0 \) for all \( t \in [t_1, \infty) \) for some \( t_1 \geq t_0 \). Suppose that there exist \( H \in \mathcal{P}, \rho \in C^1([t_0, \infty), (0, \infty)) \) and \( g \in C^1([t_0, \infty), \mathbb{R}) \) such that

\[
\limsup_{t \to \infty} \int_{l}^{t} \left[ H(s, l) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_1\alpha_2 p(s) + \mu_1 r(s)]} \Phi_1^2(s, l) \right] \, ds > 0 \quad (2.6)
\]

and

\[
\limsup_{t \to \infty} \int_{l}^{t} \left[ H(t, s) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_1\alpha_2 p(s) + \mu_1 r(s)]} \Phi_2^2(t, s) \right] \, ds > 0 \quad (2.7)
\]

for every \( l \in [t_1, \infty) \), where \( \phi, \Phi_1, \) and \( \Phi_2 \) are as in (2.2). Then (1.1) is oscillatory.

**Proof.** Suppose that \( x(t) \neq 0 \) for all \( t \in [t_2, \infty) \) for some \( t_2 \geq t_1 \). Set \( l = a \geq t_1 \) in (2.6). Clearly, we see from (2.6) that there exists \( c > a \) such that

\[
\int_{a}^{c} \left[ H(s, a) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_1\alpha_2 p(s) + \mu_1 r(s)]} \Phi_1^2(s, a) \right] \, ds > 0. \quad (2.8)
\]

Similarly, setting \( l = \alpha \geq t_2 \) in (2.7) it follows that there exists \( b > c \) such that

\[
\int_{c}^{b} \left[ H(b, s) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_1\alpha_2 p(s) + \mu_1 r(s)]} \Phi_2^2(b, s) \right] \, ds > 0. \quad (2.9)
\]
From (2.8) and (2.9) we see that (2.1) is satisfied. Therefore, in view of Corollary 2.1, we may conclude that (1.1) is oscillatory. □

**Remark 2.1.** If the inequality in (C3) is actually an equality, then the sign condition \( p(t) \geq 0 \) can be replaced by

\[
\alpha_1 \alpha_2 p(t) + \mu_1 r(t) > 0
\]

in the above theorems.

In the next theorems of this section we substitute (C3) by

\((C4)\)  \( k_1(u, v) = v k_2(u, v) \) for all \((u, v) \in \mathbb{R}^2\).

In fact, condition (C4) suffices to eliminate the sign condition on \( p \) as well.

**Theorem 2.3.** Let \((C1), (C2), \) and \((C4)\) hold, \((a, b) \subset [t_0, \infty)\) and \(c \in (a, b)\). Suppose that there exist \( H \in \mathcal{P}, \rho \in C^1([t_0, \infty), (0, \infty))\) and \( g \in C^1([t_0, \infty), \mathbb{R})\) such that

\[
\frac{1}{H(c, a)} \int_a^c \left[ H(s, a) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r(s)}{4 \mu_1} \Phi_1(s, a) \right] ds + \frac{1}{H(b, c)} \int_c^b \left[ H(b, s) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r(s)}{4 \mu_1} \Phi_2(b, s) \right] ds > 0,
\]

(2.10)

where

\[
\phi(s) = q(s) - \frac{\rho(s)}{r(s)} g(s) + \frac{\mu_1}{\alpha_1 r(s)} g^2(s) - g'(s);
\]

\[
\Phi_1(s, a) = h_1(s, a) + \left[ \frac{\rho'(s)}{\rho(s)} - \frac{\rho(s)}{r(s)} + \frac{2 \mu_1 g(s)}{\alpha_1 r(s)} \right] \sqrt{H(s, a)};
\]

\[
\Phi_2(b, s) = h_2(b, s) - \left[ \frac{\rho'(s)}{\rho(s)} - \frac{\rho(s)}{r(s)} + \frac{2 \mu_1 g(s)}{\alpha_1 r(s)} \right] \sqrt{H(b, s)}.
\]

(2.11)

Then every solution of (1.1) has a zero in \((a, b)\).

**Proof.** Otherwise, \( x(t) \neq 0 \) for all \( t \in (a, b) \). Define

\[
w(t) = \rho(t) \left[ \frac{r(t) k_1(x(t), x'(t))}{f(x(t))} + g(t) \right], \quad t \in (a, b).
\]

(2.12)

In view of (1.1), it follows from (2.12) that

\[
w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) \frac{k_2(x(t), x'(t)) x'(t)}{f(x(t))} - \rho(t) q(t) - \rho(t) r(t) k_1(x(t), x'(t)) \frac{f'(x(t)) x'(t)}{f^2(x(t))} + \rho(t) g'(t).
\]

(2.13)
Using (C1), (C2), and (C4), we see from (2.13) that
\[ w'(t) \leq -\rho(t)\phi(t) + \left[ \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} + \frac{2\mu_1 g(t)}{\varphi_1 r(t)} \right] w(t) - \frac{\mu_1}{\varphi_1 \rho(t) r(t)} w^2(t). \]

It follows from the last inequality that (1.2) holds with
\[ A_0(t) = \rho(t)\phi(t); \]
\[ A_1(t) = \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} + \frac{2\mu_1 g(t)}{\varphi_1 r(t)}; \]
\[ A_2(t) = \frac{\mu_1}{\varphi_1 \rho(t) r(t)}. \]

Applying Lemma 1.1 we see that (1.3) holds. But this contradicts (2.10). □

**Corollary 2.2.** If for a given \( T \geq t_0 \) there exists an interval \((a, b) \subset [T, \infty)\) for which the conditions of the above theorem are satisfied, then (1.1) is oscillatory.

**Theorem 2.4.** Let (C1), (C2), and (C4) be satisfied. If there exist \( H \in \mathcal{P}, \rho \in C^1([t_0, \infty), (0, \infty)) \) and \( g \in C^1([t_0, \infty), \mathbb{R}) \) such that
\[
\limsup_{t \to \infty} \int_{l}^{t} \left[ H(s, l) \rho(s) \phi(s) - \frac{\varphi_1 \rho(s) r(s)}{4\mu_1} \Phi_1^2(s, l) \right] ds > 0 \tag{2.14}
\]
and
\[
\limsup_{t \to \infty} \int_{l}^{t} \left[ H(t, s) \rho(s) \phi(s) - \frac{\varphi_1 \rho(s) r(s)}{4\mu_1} \Phi_2^2(t, s) \right] ds > 0, \tag{2.15}
\]
for all \( l \geq t_0 \), where \( \phi, \Phi_1, \) and \( \Phi_2 \) are as in (2.11), then (1.1) is oscillatory.

**Remark 2.2.** If \( k_1(u, v) = v, \ k_2(u, v) \equiv 1, \) and \( \rho(t) = \exp[-2\mu_1 \int g(t)/r(t) \, dt] \) then Theorem 2.3, Corollary 2.2, and Theorem 2.4 reduce to Theorem 2.1, Theorem 2.2, and Theorem 2.3, respectively, given in [16]. Further, by letting \( k_1(u, v) = v, \ p(t) \equiv 0, \ g(t) \equiv 0, \) and \( f(x) = x, \) we also recover Theorem 2.4 and Corollary 2.4 of Kong [11] from Corollary 2.2 and Theorem 2.4, respectively.

3. \( f(x) \) is not necessarily differentiable

We shall make use of the following conditions:

\[ C(5) \quad f(x)/x \geq \mu_2 \text{ for some positive constant } \mu_2 \text{ and all } x \in \mathbb{R} \setminus \{0\}; \]
\[ C(6) \quad u v k_2(u, v) \geq \varphi_3 k_1^2(u, v) \text{ for some } \varphi_3 \geq 0 \text{ and all } (u, v) \in \mathbb{R}^2. \]

**Theorem 3.1.** Let (C2), (C5), and (C6) hold. Suppose that there exists an interval \((a, b) \subset [t_0, \infty)\) such that \( p(t) \geq 0 \) and \( q(t) \geq 0 \) for all \( t \in (a, b), \) and that there exist \( H \in \mathcal{P}, \)
\( \rho \in C^1([t_0, \infty), (0, \infty)) \) and \( g \in C^1([t_0, \infty), \mathbb{R}) \) such that

\[
\frac{1}{H(c,a)} \int_a^c \left[ H(s,a) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_2 p(s) + r(s)]} \Phi_1^2(s,a) \right] ds \\
+ \frac{1}{H(b,c)} \int_c^b \left[ H(b,s) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_2 p(s) + r(s)]} \Phi_2^2(b,s) \right] ds > 0, \tag{3.1}
\]

where

\[
\phi(s) = \mu_2 q(s) + g^2(s) \frac{\alpha_1 \alpha_2 p(s) + r(s)}{\alpha_1 r^2(s)} - g'(s);
\]

\[
\Phi_1(s,a) = h_1(s,a) + \left[ \frac{\rho'(s)}{\rho(s)} + 2g(s) \frac{\alpha_1 \alpha_2 p(s) + r(s)}{\alpha_1 r^2(s)} \right] \sqrt{H(s,a)};
\]

\[
\Phi_2(b,s) = h_2(b,s) - \left[ \frac{\rho'(s)}{\rho(s)} + 2g(s) \frac{\alpha_1 \alpha_2 p(s) + r(s)}{\alpha_1 r^2(s)} \right] \sqrt{H(b,s)}.
\tag{3.2}
\]

Then every solution of (1.1) has a zero in \((a, b)\).

**Proof.** Otherwise, \( x(t) \neq 0 \) for all \( t \in (a, b) \). Define

\[
w(t) = \rho(t) \left[ \frac{r(t) k_1(x(t), x'(t))}{x(t)} + g(t) \right], \quad t \in (a, b).
\tag{3.3}
\]

In view of (1.1), it follows from (2.3) that

\[
w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) \frac{k_2(x(t), x'(t)) x'(t)}{x(t)} \\
- \rho(t) q(t) \frac{f(x(t))}{x(t)} - \rho(t) r(t) k_1(x(t), x'(t)) \frac{x'(t)}{x^2(t)} + \rho(t) g'(t).
\tag{3.4}
\]

Using (C2), (C5), and (C6) in (3.4) result in

\[
w'(t) \leq -\rho(t) \phi(t) + \left[ \frac{\rho'(t)}{\rho(t)} + 2g(t) \frac{\alpha_1 \alpha_2 p(t) + r(t)}{\alpha_1 r^2(t)} \right] w(t) \\
- \frac{\alpha_1 \alpha_2 p(t) + r(t)}{\alpha_1 \rho(t) r^2(t)} w^2(t).
\tag{3.5}
\]

Comparing with (1.2) we have

\[
A_0(t) = \rho(t) \phi(t);
\]

\[
A_1(t) = \frac{\rho'(t)}{\rho(t)} + 2g(t) \frac{\alpha_1 \alpha_2 p(t) + r(t)}{\alpha_1 r^2(t)};
\]

\[
A_2(t) = \frac{\alpha_1 \alpha_2 p(t) + r(t)}{\alpha_1 \rho(t) r^2(t)}.
\]

Applying Lemma 1.1 we see that (1.3) holds, which clearly contradicts (3.1). \( \square \)

**Corollary 3.1.** If for a given \( T \geq t_0 \) there exists an interval \((a, b) \subset [T, \infty)\) for which the conditions of the above theorem are satisfied, then (1.1) is oscillatory.
Theorem 3.2. Let (C2), (C5), and (C6) hold, and \( p(t) \geq 0 \) and \( q(t) \geq 0 \) for all \( t \in [t_1, \infty) \) for some \( t_1 \geq t_0 \). If there exist \( H \in \mathcal{P}, \rho \in C^1([t_0, \infty), (0, \infty)) \) and \( g \in C^1([t_0, \infty), \mathbb{R}) \) such that

\[
\limsup_{t \to \infty} \int_l^t \left[ H(s, l) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_2 \rho(s) + r(s)]} \Phi_1^2(s, l) \right] ds ds > 0
\]

and

\[
\limsup_{t \to \infty} \int_l^t \left[ H(t, s) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_2 \rho(s) + r(s)]} \Phi_2^2(t, s) \right] ds > 0
\]

for every \( l \in [t_1, \infty) \), where \( \phi, \Phi_1, \) and \( \Phi_2 \) are as in (3.2), then (1.1) is oscillatory.

Remark 3.1. If (C6) holds with equality, then the sign condition \( p(t) \geq 0 \) can be replaced by

\[
\alpha_2 \rho(t) + r(t) > 0
\]

in the above theorems.

As in the previous section, we may replace condition (C6) and allow \( p \) to change sign. Specifically we have the following theorems.

Theorem 3.3. Let (C2), (C4), and (C5) hold. Suppose that there exists an interval \( (a, b) \subset [t_0, \infty) \) such that \( q(t) \geq 0 \) for all \( t \in (a, b) \), and that there exist \( c \in (a, b), H \in \mathcal{P}, \rho \in C^1([t_0, \infty), (0, \infty)) \) and \( g \in C^1([t_0, \infty), \mathbb{R}) \) such that

\[
\frac{1}{H(c, a)} \int_a^c \left[ H(s, a) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_2 \rho(s) + r(s)]} \Phi_1^2(s, a) \right] ds + \frac{1}{H(b, c)} \int_c^b \left[ H(b, s) \rho(s) \phi(s) - \frac{\alpha_1 \rho(s) r^2(s)}{4[\alpha_2 \rho(s) + r(s)]} \Phi_2^2(b, s) \right] ds > 0, \tag{3.6}
\]

where

\[
\phi(s) = \mu_2 g(s) - \frac{p(s)}{r(s)} g(s) + \frac{1}{\alpha_1 r(s)} g^2(s) - g'(s);
\]

\[
\Phi_1(s, a) = h_1(s, a) + \left[ \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)} + \frac{2g(s)}{\alpha_1 r(s)} \right] \sqrt{H(s, a)};
\]

\[
\Phi_2(b, s) = h_2(b, s) - \left[ \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)} + \frac{2g(s)}{\alpha_1 r(s)} \right] \sqrt{H(b, s)}.
\tag{3.7}
\]

Then every solution of (1.1) has a zero in \( (a, b) \).

Proof. Otherwise, \( x(t) \neq 0 \) for all \( t \in (a, b) \). Define

\[
w(t) = \rho(t) \left[ \frac{r(t) k_1(x(t), x'(t))}{x(t)} + g(t) \right], \quad t \in (a, b). \tag{3.8}
\]
In view of (1.1), it follows from (3.8) that

$$w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\rho(t) p(t) k_2(x(t), x'(t)) x'(t)}{x(t)}$$

$$- \rho(t) q(t) \frac{f(x(t))}{x(t)} - \rho(t) r(t) k_1(x(t), x'(t)) \frac{x'(t)}{x^2(t)} + \rho(t) g'(t).$$

Using (C2), (C4), and (C5), we see that

$$w'(t) \leq - \frac{\rho(t) p_1(t)}{r(t)} + \left[ \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} + \frac{2 g(t)}{x_1 r(t)} \right] w(t) - \frac{1}{x_1 \rho(t) r(t)} w^2(t).$$

Comparing with (1.2) we have

$$A_0(t) = \rho(t) \phi(t);$$
$$A_1(t) = \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} + \frac{2 g(t)}{x_1 r(t)};$$
$$A_2(t) = \frac{1}{x_1 \rho(t) r(t)}.$$

We apply Lemma 1.1 to get a contradiction with (3.6). □

**Corollary 3.2.** If for a given \( T \geq t_0 \) there exists an interval \((a, b) \subset [T, \infty)\) for which the conditions of the above theorem are satisfied, then (1.1) is oscillatory.

**Theorem 3.4.** Let (C2), (C4), and (C5) be satisfied, and \( q(t) \geq 0 \) for all \( t \in [t_1, \infty) \) for some \( t_1 \geq t_0 \). If there exist \( H \in \mathcal{P}, \rho \in C^1([t_0, \infty), (0, \infty)) \) and \( g \in C^1([t_0, \infty), \mathbb{R}) \) such that

$$\limsup_{t \to \infty} \int_t^\infty \left[ H(s, l) \rho(s) \phi(s) - \frac{x_1 \rho(s) r(s)}{4} \Phi_1^2(s, l) \right] ds > 0$$

(3.10)

and

$$\limsup_{t \to \infty} \int_t^\infty \left[ H(t, s) \rho(s) \phi(s) - \frac{x_1 \rho(s) r(s)}{4} \Phi_2^2(t, s) \right] ds > 0,$$

(3.11)

for all \( l \geq t_1 \), where \( \phi, \Phi_1, \) and \( \Phi_2 \) are as in (3.7), then (1.1) is oscillatory.

**Remark 3.2.** If \( f(x) = x \) then the sign condition \( q(t) \geq 0 \) is not necessary.

**Remark 3.3.** If \( k_1(u, v) = v \) and \( p(t) \equiv 0 \), Theorem 3.3, Corollary 3.2, Theorem 3.4, reduce to Theorem 3.1, Theorem 3.2, and Theorem 3.3, respectively, given in [15,17].
4. The perturbed case

With regard to Eq. (1.1) let us consider

\[(r(t)k_1(x, x'))' + p(t)k_2(x, x')x' + q(t)f(x) = h(t, x, x'), \quad t \geq t_0,\]

(4.1)

where \(h(t, u, v)\) is a continuous function.

Suppose that there exist sequences \((a_n, b_n)\) and \((\tilde{a}_n, \tilde{b}_n)\), \(\lim a_n = \lim \tilde{a}_n = \infty\), such that

\[uh(t, u, v) \geq 0, \quad \text{if } u > 0 \text{ and } t \in (a_n, b_n);\]

(4.2)

\[uh(t, u, v) \geq 0, \quad \text{if } u < 0 \text{ and } t \in (\tilde{a}_n, \tilde{b}_n).\]

(4.3)

We shall also assume that

\[xf(x) > 0, \quad \text{if } x \neq 0.\]

(4.4)

**Theorem 4.1.** Suppose that (4.2)–(4.4) are satisfied. If the conditions of either Theorem 2.1 or Theorem 2.3 or Theorem 3.1 or Theorem 3.3 hold with \((a, b) = (a_n, b_n)\), \(c = c_n \in (a_n, b_n)\), and also with \((a, b) = (\tilde{a}_n, \tilde{b}_n)\), \(c = \tilde{c}_n \in (\tilde{a}_n, \tilde{b}_n)\) then (4.1) is oscillatory.

**Proof.** We sketch the proof in case the conditions of Theorem 2.1 hold. The other cases is similar. We assume that \(x(t)\) is nonoscillatory. If \(x(t)\) is eventually positive, then we pick intervals \((a_n, b_n)\) with \(n\) sufficiently large. Since \(h(t, x(t), x'(t)) / x(t) \geq 0\) for all \(t \in (a_n, b_n)\) we observe that inequality (2.5) remains the same, and so the rest of the proof is similar to that of Theorem 2.1. Next, we should consider the case \(x(t) < 0\) eventually. Choosing the sequence \((\tilde{a}_n, \tilde{b}_n)\) in this case and noting that \(h(t, x(t), x'(t)) / x(t) \geq 0\) for all \(t \in (\tilde{a}_n, \tilde{b}_n)\), proceeding as before we arrive at a contradiction. 

It is clear that there is a large class of functions with the above properties. The following are only a few examples of such functions:

\[
\begin{align*}
\frac{\sin t}{t}, & \quad |u|^{\alpha-1}u \quad (\alpha > 0), \quad \cos t \frac{|u|^{x-1}u}{2 + \cos(u^2v)}, \quad \frac{u^2 \sin t}{u^2 + tv^4},
\end{align*}
\]

**Remark 4.1.** The problem when \(h(t, u, v) = h_0(t)\) with \(h_0\) being nonoscillatory remains to be open.

5. Examples

The oscillation of various special cases of (1.1) or (4.1) has been investigated by many authors. Our results, however, are still new even in some particular cases because we allow the coefficients to change sign. The arbitrariness of functions \(\rho\) and \(g\) and the class of functions \(H\) provide more flexibilities for deriving several oscillation criteria. Among many
other possibilities the following choices of functions \( H(t, s) \) have been used extensively in the literature:

\[
H(t, s) = (t - s)\lambda, \quad (\lambda \geq 1);
\]

\[
H(t, s) = [R(t) - R(s)]\lambda, \quad R(t) = \int_{t_0}^{t} \frac{1}{r(s)} \, ds, \quad (\lambda > 1);
\]

\[
H(t, s) = \ln \frac{Q(t)}{Q(s)}, \quad Q(t) = \int_{t_0}^{t} \frac{1}{r(s)} \, ds;
\]

\[
H(t, s) = \left[ \int_{s}^{t} \frac{1}{\psi(u)} \, du \right]^{\lambda/2}, \quad (\lambda > 1), \quad \psi \in C([t_0, \infty), (0, \infty)), \quad \int_{t_0}^{\infty} \frac{1}{\psi(u)} \, du = \infty.
\]

In this section we provide some examples to illustrate the results obtained in this paper. The examples are given with respect to (1.1) as they can be modified trivially for (4.1).

Example 5.1. Let (C2) and (C4) be satisfied, we may take for instance \( k_1(u, v) \) as follows:

\[
\frac{u^2 v^3}{1 + u^2 v}, \quad \frac{u^2 v^3}{1 + u^2}, \quad \frac{v}{1 + u^2}, \quad \frac{v \cos^2 u}{1 + u^2}, \quad (x_1 = 1);
\]

Fix \( f(x) \) so that (C1) holds, e.g., \( f(x) = x + x^3 \); Let

\[
q(t) = \begin{cases}
2t - (6n - 7/2)\pi, & (6n - 4)\pi \leq t \leq (6n - 7/2)\pi, \\
t, & (6n - 7/2)\pi < t \leq (6n - 3)\pi \\
q_0(t), & (6n - 3)\pi < t \leq (6n + 2)\pi
\end{cases}
\]

for \( n \in \{1, 2, \ldots\} \), where \( q_0 \) is any continuous function which makes \( q \) a continuous function. It is worth mentioning that by a suitable choice of \( q_0(t) \) we can make

\[
\int_{t_0}^{\infty} q(t) \, dt = -\infty,
\]

meaning that the results of this paper are applicable for such extreme cases, and hence improve Kamenev’s type oscillation criteria [10] and extend them to (1.1) and (4.1).

Consider the nonlinear differential equation

\[
(k_1(x, x'))' + \sin t k_1(x, x') + q(t) f(x) = 0. \tag{5.1}
\]

By taking \( H(t, s) = (t - s)^2, \rho(t) = 1, \ g(t) = 0, \ a = (6n - 4)\pi, \ c = (6n - 7/2)\pi, \ b = (6n - 3)\pi \) we can easily evaluate the integral in (2.10) as

\[
\frac{96\pi^3 n - 204 - 2\pi^2 - 57\pi^3}{48\pi} > 0, \quad n \geq 1.
\]

Thus in view of Corollary 2.2 we may conclude that (5.1) is oscillatory. In fact, if \( x(t) \) is a solution of (5.1) then there exists a sequence \( \{t_n\} \), \( (6n - 4)\pi < t_n < (6n + 2)\pi \) such that \( x(t_n) = 0 \).

Example 5.2. Consider

\[
((1 + \sin^2 t)x')' - 3 \sin t \cos t x' + \frac{1}{1 + \cos^4 t} x(1 + x^4) = 0. \tag{5.2}
\]
so that \( k_1(u, v) = v, k_2(u, v) \equiv 1, \alpha_1 = 1, f(x) = x + x^5, \mu_1 = 1, r(t) = 1 + \sin^2 t, q(t) = 1/(1 + \cos^4 t) \).

If \( H(t, s) = (t - s)^2, \rho(t) = t^2, g(t) = -r(t)/t \), then after some tedious computations one can verify that (2.14) and (2.15) are satisfied. Applying Theorem 2.4, we see that (5.2) must be oscillatory. In fact, \( x(t) = \cos t \) is an oscillatory solution.

**Example 5.3.** Consider

\[
((3 + \sin t)k_1(x, x')\)' + (\delta + \sin t)k_2(x, x')x' + \cos^2 tf(x) = 0,
\]

(5.3)

where \( \delta \geq 1 \) and

\[
k_1(u, v) = \frac{u^2v}{1 + u^2}, \quad k_2(u, v) = \frac{u^3v(1 + u^2 + v^2)}{(1 + u^2)^2}, \quad (\alpha_1 = \alpha_3 = 1),
\]

\[
f(x) = x(2 + \cos x), \quad (\mu_2 = 1).
\]

Note that the function \( f \) satisfies \( f(x)/x \geq 1 \) for all \( x \neq 0 \), but there is no \( L \) for which \( f'(x) \geq L \) for all \( x \in \mathbb{R}\setminus\{0\} \).

Letting \( H(t, s) = (t - s)^2, \rho(t) = t, \) and \( g(t) \equiv 0 \) we see that all the conditions of Theorem 3.2 hold, and hence (5.3) is oscillatory.

**Example 5.4.** Consider

\[
(k_1(x, x')\)' + \cos tk_2(x, x')x' + \sin tf(x) = 0.
\]

(5.4)

Let \( H(t, s) = (t - s)^2, \rho(t) \equiv 1, g(t) \equiv 0, a = (2n + 3/2)\pi, c = (2n + 5/2)\pi, \) and \( b = (2n + 7/2)\pi \).

If \( k_1(u, v) = v, k_2(u, v) \equiv 1, \) and \( f(x) = x \), then it is known that (5.4) is oscillatory, see [13]. Applying Corollary 2.1 or Corollary 3.1 we recover this result.

Moreover, if \( k_1, k_2, \) and \( f \) satisfy either conditions of Theorem 2.1 or Theorem 3.1, e.g., if

\[
k_1(u, v) = \frac{u^2v}{1 + u^2}, \quad k_2(u, v) = \frac{u^3v(1 + u^2 + v^2)}{(1 + u^2)^2}, \quad (\alpha_1 = \alpha_2 = \alpha_3 = 1),
\]

\[
f(x) = x(2 + \cos x) \quad \text{(or } f(x) = x(1 + x^3)\text{)}, \quad (\mu_1 = \mu_2 = 1)
\]

then we see from Corollary 2.1 (or Corollary 3.1) that (5.4) is oscillatory. Indeed, the integrals in (2.1) and (3.1) coincide and give \( 33/3\pi - \pi/12 > 0 \).

**References**

