Abstract—Sufficient conditions are established for the existence of positive solutions and oscillation of all bounded solutions of the neutral difference equation

\[ \Delta^n[x_n - cx_{n-1}] + q_n f(x_{n-k}) = h_n, \quad n \geq n_0, \]

where \( \Delta \) is the forward difference operator \( \Delta x_n = x_{n+1} - x_n \), \( l \) and \( k \) are integers, \( c \neq \pm 1 \) is a real number, and \( \{q_n\} \) and \( \{h_n\} \) are real sequences. It is also shown that some of these sufficient conditions are necessary.

Keywords—Difference equation, Neutral equation, Positive solution, Oscillation.

1. INTRODUCTION

In the past ten years, there has been increasing interest in studying the oscillatory and nonoscillatory behavior of solutions of difference equations. Much of the research on the subject, however, has been restricted to oscillation and nonoscillation of first-order and second-order equations (see [1–8] and the references cited therein). For instance, Lalli et al. [5] studied the first-order neutral difference equation

\[ \Delta[x_n - cx_{n-1}] + q_n x_{n-k} = 0 \]

and proved, among the others, that if \( c > 1 \), \( q_n \geq 0 \), and \( \sum_{n=1}^{\infty} q_n = \infty \), then every nonoscillatory solution tends to \( \infty \) or \( -\infty \) as \( n \to \infty \), or equivalently, every bounded solution is oscillatory.

Recently, Lalli [6] considered a more general equation, but still of first order, and proved, under similar conditions, that the bounded solutions are oscillatory.

It should be noted that almost all the results concerning the oscillatory and nonoscillatory behavior of difference equations are obtained as the discrete analogues of those for differential equations. The ideas behind the analogues are similar, but quite different due to the discrete nature. Motivation of this study also stems from a work of Zhang and Yu [9], who studied the neutral equation

\[ [x(t) - cx(t - \tau)]'' + p(t)x(g(t)) = 0 \]

and showed that, under the conditions \( p(t) \leq 0 \), \( c > 1 \), and \( \tau > 0 \), it has a bounded positive solution if and only if \( - \int_{0}^{\infty} tp(t)\ dt < \infty \), and if \( 0 < c < 1 \), \( \tau > 0 \) and \( \int_{0}^{\infty} |p(t)|\ dt < \infty \), then there is a bounded positive solution.
In this paper, we consider the \( p \)-th-order neutral difference equation of the form
\[
\Delta^p[x_n - cx_{n-l}] + q_n f(x_{n-k}) = h_n,
\] 
(1)
where \( n \in N(n_0) = \{n_0, n_0 + 1, \ldots \} \), \( n_0 \) a fixed positive integer. We assume that \( l \) and \( k \) are integers with \( l > 0 \), \( x, q, h, f \) are real valued functions with \( x, q, h : N(n_0) \to R = (-\infty, \infty) \) and \( f : R \to R \). It is also assumed that \( f \) is continuous.

By a solution of equation (1), we mean a real sequence \( \{x_n\} \) satisfying equation (1), so that
\[
\sup_{n \geq m} |x_n| \neq 0 \text{ for any } m \in N(n_0).
\]
We always assume that such solutions of equation (1) exist. A solution of equation (1) is called oscillatory if there is no end of \( n_1 \) and \( n_2 \) \( (n_1 < n_2) \) in \( N(n_0) \) such that \( x_{n_1} x_{n_2} \leq 0 \); otherwise, it is called nonoscillatory. Clearly, a nonoscillatory solution of equation (1) must be eventually of fixed sign.

Our purpose here is to first find sufficient conditions for the existence of positive solutions and oscillation of bounded solutions of equation (1), and then show that some of these conditions are also necessary. In Section 2, we study equation (1) for all values of \( c \) except \( c = \pm 1 \) and find sufficient conditions under which equation (1) has a bounded positive solution. Section 3 contains necessary and sufficient conditions for oscillation of all bounded solutions of equation (1) when \( c = 0 \) and \( c > 1 \). In the special case of \( p = 2 \), \( f(x) = x \), and \( h_n = 0 \), our results may be considered as the discrete analogues of some of those of Zhang and Yu [9]. For more results regarding oscillation and asymptotic behavior of higher-order difference equations, we refer in particular to [10-13].

Before we state and prove our results, we should mention that our equation is quite general and therefore the results of this paper even in some special cases complement and generalize many results in the literature, including some of those of He [3], Lalli et al. [5,6], Agarwal [10,11], and Zafer [12,13].

### 2. POSITIVE SOLUTIONS

In what follows, \( n^{(s)} \) will denote the usual factorial function; that is, \( n^{(0)} = 1 \) and \( n^{(s)} = n(n-1) \cdots (n-s+1) \).

**Theorem 1.** Let \( c = 0 \) and let
\[
\sum_{n=0}^{\infty} n^{(p-1)} |q_n| < \infty \tag{2}
\]
and
\[
\sum_{n=0}^{\infty} n^{(p-1)} |h_n| < \infty. \tag{3}
\]
Then, equation (1) has a bounded positive solution \( \{x_n\} \).

**Proof.** Let \( a \) be a positive real number and \( M = \max\{|f(x)| : a \leq x \leq 2a\} \). Choose \( n_1 \) large enough so that for \( n \geq n_1 \), \( n \geq n_0 - k \),
\[
\sum_{s=n}^{\infty} s^{(p-1)} |q_s| < \frac{(p-1) a}{4M}, \tag{4}
\]
and
\[
\sum_{s=n}^{\infty} s^{(p-1)} |h_s| < \frac{(p-1) a}{4}. \tag{5}
\]
Introduce the Banach space \( Y \) of all functions \( x : N(n_0) \to R \) such that
\[
\|x\| = \sup_{n \in N(n_0)} |x_n|.
\]
Define
\[ X = \{ x \in Y : a \leq x_n \leq 2a, n \in N(n_0) \}. \]
Clearly \( X \) is a bounded, convex, and closed subset of \( Y \). We also define an operator \( T \) on \( X \) by
\[
T_x_n = \frac{3a}{2} + \frac{(-1)^{p-1}}{(p-1)!} \sum_{s=n}^{\infty} (s+p-1-n)^{(p-1)} q_s f(x_{s-k})
\]
\[
+ \frac{(-1)^{p}}{(p-1)!} \sum_{s=n}^{\infty} (s+p-1-n)^{(p-1)} h_s, \quad n \in N(n_1)
\]
\[= T x_{n_1}, \quad n_0 \leq n \leq n_1. \]

The mapping \( T \) satisfies the assumptions of the Schauder's fixed-point theorem. Namely, it satisfies the following.

(a) \( T \) maps \( X \) into itself. In fact, if \( x \in X \), then in view of (4) and (5), it follows that
\[
Tx_n \leq \frac{3a}{2} + \frac{1}{(p-1)!} M \frac{a(p-1)!}{4M} + \frac{1}{(p-1)!} \frac{a(p-1)!}{4} = 2a
\]
and
\[
Tx_n \geq \frac{3a}{2} - \frac{1}{(p-1)!} M \frac{a(p-1)!}{4M} - \frac{1}{(p-1)!} \frac{a(p-1)!}{4} = a.
\]
Therefore, \( TX \subseteq X \).

(b) \( T \) is continuous. Indeed, let \( \{ x^i_n \} \) be a Cauchy sequence in \( X \), and let \( \lim_{i \to \infty} \| x_i - x \| = 0 \). Because \( X \) is closed, \( x \in X \). Now
\[
|Tx^i_n - Tx_n| \leq \frac{1}{(p-1)!} \sum_{s=n}^{\infty} (s+p-1-n)^{(p-1)} q_s |f(x^i_{s-k}) - f(x_{s-k})|.
\]
Since
\[
\sum_{s=n}^{\infty} (s+p-1-n)^{(p-1)} q_s |f(x^i_{s-k}) - f(x_{s-k})| \leq 2M \sum_{s=n}^{\infty} s^{(p-1)} q_s < \infty
\]
and \( f \) is continuous, it follows from (6) that
\[
\lim_{i \to \infty} \| Tx^i_n - Tx_n \| = 0.
\]
This means that \( T \) is continuous.

(c) \( TX \) is precompact. As pointed out by Xue [3], it is enough to show that \( TX \) is uniformly Cauchy. So let \( x \in X \) and \( n, m \in N(n_0) \). Then, for \( n > m \),
\[
|Tx_n - Tx_m| \leq \sum_{s=n}^{\infty} (s+p-1-n)^{(p-1)} q_s |f(x_{s-k})|
\]
\[+ \sum_{s=m}^{\infty} (s+p-1-m)^{(p-1)} q_s f(x_{s-k})|
\[+ \sum_{s=n}^{\infty} (s+p-1-n)^{(p-1)} h_s + \sum_{s=m}^{\infty} (s+p-1-m)^{(p-1)} |h_s|
\[\leq 2M \sum_{s=m}^{\infty} s^{(p-1)} q_s + 2 \sum_{s=m}^{\infty} s^{(p-1)} |h_s|.
\]
Using (2) and (3), it is clear that for a given $\varepsilon > 0$, there exists an integer $n_2 \in N(n_1)$ such that for all $x \in X$ and $n, m \in N(n_2)$, 

$$|Tx_n - Tx_m| < \varepsilon.$$ 

This shows that $TX$ is uniformly Cauchy, and hence, $TX$ is precompact.

According to Schauder’s fixed-point theorem, there exists an $x \in X$ such that $Tx = x$. This $x$ is a bounded positive solution of equation (1). The proof is now complete.

In the following theorems, we assume that $f$ satisfies a Lipschitz condition on the given interval; i.e., there is a number $L$ such that

$$|f(x) - f(y)| \leq L|x - y|, \quad \text{for all } x, y \in [a, b], \quad (7)$$

where $[a, b]$ is either $[c, 2c]$ or $[-c, -2c]$ or $[1/2c, 1/c]$ or $[-1/2c, -1/c]$, depending on $c$.

**Theorem 2.** Suppose that (2) and (3) hold and $c > 1$, and let (7) be satisfied with $[a, b] = [c, 2c]$. Then, equation (1) has a bounded positive solution $\{x_n\}$.

**Proof.** Let $n_1$ be large enough so that for $n > n_1$, $n > \max\{n_0 - l, n_0 - l + k\}$,

$$\sum_{s=n+1}^{\infty} s^{(p-1)}|q_s| < \frac{\alpha}{4M}, \quad (8)$$

and

$$\sum_{s=n+1}^{\infty} s^{(p-1)}|h_s| < \frac{\beta}{4}, \quad (9)$$

where $\alpha = (p-1)!/(c-1)/2$, $\beta = (p-1)!/(c-1)$, and $M = \max\{K, L\}$, $K = \sup_{x \in [a, b]}(|f(x)|/|x|)$.

We introduce the Banach space $Y$ of all functions $x : N(n_0) \to R$ such that

$$\|x\| = \sup_{n \in N(n_0)} |x_n|.$$

Define

$$X = \{x \in Y : a \leq x_n \leq b, n \in N(n_0)\}.$$ 

Clearly, $X$ is a bounded, convex, and closed subset of $Y$. We also define an operator $T$ on $X$ by

$$Tx_n = \frac{3(c-1)}{2} + \frac{1}{c}x_{n+l}$$

$$+ \frac{(-1)^p}{c(p-1)!} \sum_{s=n+1}^{\infty} (s + p - 1 - n - l)^{(p-1)}q_s f(x_{s-k})$$

$$+ \frac{(-1)^{p-1}}{c(p-1)!} \sum_{s=n+1}^{\infty} (s + p - 1 - n - l)^{(p-1)}h_s, \quad \text{for } n \in N(n_1)$$

$$= Tx_{n_1}, \quad \text{for } n_0 \leq n \leq n_1.$$ 

We shall show that $T$ is a contraction mapping on $X$. It is easy to see that $T$ maps $X$ into itself. In fact, if $x \in X$, then because of (8) and (9), it follows that

$$Tx_n \leq \frac{3(c-1)}{2} + \frac{1}{c} \cdot 2c + \frac{c-1}{4} + \frac{c-1}{4} = 2c$$

and

$$Tx_n \geq \frac{3(c-1)}{2} + \frac{1}{c} \cdot c - \frac{c-1}{4} - \frac{c-1}{4} = c.$$ 

Therefore, $TX \subseteq X$. 

To show that $T$ is a contraction, let $x, y \in X$. Then,

$$|Tx_n - Ty_n| \leq \frac{1}{c} |x_{n+1} - y_{n+1}|$$

$$+ \frac{1}{c(p-1)!} M \sum_{s=n+1}^{\infty} (s+p-1-n-l)^{(p-1)} |q_s| |x_{s-k} - y_{s-k}|$$

$$\leq \frac{1}{c} \|x - y\| + \frac{1}{c(p-1)!} \|x - y\| M \sum_{s=n+1}^{\infty} (s+p-1-n-l)^{(p-1)} |q_s|.$$

Clearly,

$$|Tx_n - Ty_n| \leq \frac{1}{c} \|x - y\| + \frac{c-1}{8c} \|x - y\|$$

and so

$$\|Tx - Ty\| \leq \left( \frac{1}{c} + \frac{c-1}{8c} \right) \|x - y\| < \|x - y\|.$$

Since $T$ is a contraction on $X$, it follows that there exists a fixed-point $x \in X$ such that $Tx = x$. It can easily be seen that $x$ is a bounded positive solution of equation (1). This completes the proof.

**Example 1.** Consider

$$\Delta^3(x_n - 2x_{n-2}) - \frac{2^{-n}}{(3 + 2^{-n})^5} x_n^5 = -2^{-n-3},$$

so that $p = 3, l = 2, k = 6, f(x) = x^5, q_n = -2^{-n}(1 + 2^{-n})^{-5},$ and $h_n = -2^{-n-3}$. It is clear that the conditions of Theorem 2 are satisfied, and hence, the above equation has a bounded positive solution. Indeed, $x_n = 3 + 2^{-n}$ is such a solution.

The following theorem shows that the conclusion of Theorem 2 remains valid also in the cases of $c < -1, -1 < c < 0,$ and $0 < c < 1$.

**Theorem 3.** Suppose that (2) and (3) hold. Then, equation (1) has a bounded positive solution $\{x_n\}$ in each of the following cases:

(i) $c < -1$ and (7) is satisfied with $[a, b] = [-c, -2c],$

(ii) $-1 < c < 0$ and (7) is satisfied with $[a, b] = [-1/2c, -1/c],$

(iii) $0 < c < 1$ and (7) is satisfied with $[a, b] = [1/2c, 1/c].$

**Proof.** The proof in each case can be carried out exactly as in the previous theorem if the following changes in the definitions of $\alpha, \beta,$ and $T$ are established.

If $c < -1$, let $\alpha = (p-1)!(-1-c)/2, \beta = (p-1)!c(c+1),$ and

$$Tx_n = \frac{3(1-c)}{2} + \frac{1}{c} x_{n+1} + \frac{(-1)^p}{c(p-1)!} \sum_{s=n+1}^{\infty} (s+p-1-n-l)^{(p-1)} q_s f(x_{s-k})$$

$$+ \frac{(-1)^{p-1}}{c(p-1)!} \sum_{s=n+1}^{\infty} (s+p-1-n-l)^{(p-1)} h_s, \text{ for } n \in N(n_1)$$

$$= Tx_{n_1} \text{ for } n_0 \leq n \leq n_1.$$

If $-1 < c < 0$, let $\alpha = (p-1)!(-1-c)/2, \beta = -(p-1)!/(1+c+1)/2,$ and

$$Tx_n = \frac{3(c-1)}{4c} + cx_{n+1} + \frac{(-1)^{p-1}}{(p-1)!} \sum_{s=n}^{\infty} (s+p-1-n-l)^{(p-1)} q_s f(x_{s-k})$$

$$+ \frac{(-1)^p}{(p-1)!} \sum_{s=n}^{\infty} (s+p-1-n-l)^{(p-1)} h_s, \text{ for } n \in N(n_1)$$

$$= Tx_{n_1} \text{ for } n_0 \leq n \leq n_1.$$
And finally, if $0 < c < 1$, let $\alpha = (p - 1)!/(1 - c)/2$, $\beta = (p - 1)!/(1/c - 1)/2$, and

$$Tx_n = \frac{3(1 - c)}{4c} + cx_{n-1} + \frac{(-1)^{p-1}}{(p-1)!} \sum_{s=n}^{\infty} (s + p - 1 - n)^{(p-1)} q_s f(x_{s-k})$$

$$+ \frac{(-1)^p}{(p-1)!} \sum_{s=n}^{\infty} (s + p - 1 - n)^{(p-1)} h_s, \quad \text{for } n \in N(n_1)$$

$$= Tx_{n_1}, \quad \text{for } n_0 \leq n \leq n_1.$$

**REMARK.** A close look at the proof of each theorem reveals that the assumption about the continuity of $f$ on all of $R$ is not necessary. For example, Theorem 2 holds true even if $f$ is continuous only in the interval $[c, 2c]$.

### 3. OSCILLATORY BEHAVIOR

In this section, we study the oscillation behavior of bounded solutions of equation (1) under some additional conditions. In particular, we shall assume that the sequence $\{q_n\}$ is eventually of fixed sign, $x$ and $f(x)$ has the same sign, and that there exists an oscillatory function on $N$ satisfying certain properties. It is therefore more convenient to rewrite equation (1) in the form

$$\Delta^p [x_n - cx_{n-1}] + \delta q_n f(x_{n-k}) = h_n,$$

and assume that $\delta = \pm 1$, $xf(x) > 0$ for $x \neq 0$, and that $q_n \geq 0$ with infinitely many positive terms.

We will need the following lemmas of Agarwal [10].

**LEMMA 1.** Let $\{y_n\}$ and $\{\Delta^p y_n\}$ be sequences defined on $N(n_0)$ with $y_n > 0$ and $\Delta^p y_n < 0$ on $N(n_0)$. Then there exists an integer $l$, $0 \leq l \leq p - 1$ with $p - l$ odd such that for $n \in N(n_0)$,

$$\Delta^j y_n > 0, \quad \text{for } j = 0, 1, \ldots, l,$$

$$(-1)^{l-j} \Delta^j y_n > 0, \quad \text{for } j = l + 1, \ldots, p - 1.$$

**LEMMA 2.** Let $\{y_n\}$ and $\{\Delta^p y_n\}$ be sequences defined on $N(n_0)$ with $y_n > 0$ and $\Delta^p y_n > 0$ on $N(n_0)$. Then, for $n \in N(n_0)$, either

$$\Delta^j y_n > 0, \quad j = 1, \ldots, p$$

or there exists an integer $l$, $0 \leq l \leq n - 2$ with $p - l$ even such that for $n \in N(n_0)$,

$$\Delta^j y_n > 0, \quad \text{for } j = 0, 1, \ldots, l,$$

$$(-1)^{l-j} \Delta^j y_n > 0, \quad \text{for } j = l + 1, \ldots, p - 1.$$

**LEMMA 3.** Let $\{y_n\}$ be some sequence defined on $N(n_0)$. Then,

$$\sum_{s=n_1}^{n-1} s^{(p-1)} \Delta^p y_s = \sum_{k=1}^{p} (-1)^{k+1} s^{(p-1)} \Delta^{p-k} y_{s+k-1} |_{s=n_1}.$$

We are now ready to prove our oscillation theorems for equation (10) when $c = 0$ and $c > 1$. If $-1 < c < 0$, similar results may be found in [13].
THEOREM 4. Let \( c = 0 \) and let \( \phi : N \to R \) be an oscillatory function such that \( \Delta^p \phi_n = h_n \) and \( \lim_{n \to \infty} \Delta^j \phi_n = 0 \) for \( j = 1, 2, \ldots, p - 1 \). If
\[
\sum_{s=0}^{\infty} s^{(p-1)} q_s = \infty,
\] (14)
then every bounded solution \( \{x_n\} \) of equation (10) is oscillatory when \( (-1)^{p-1} = 1 \), and is either oscillatory or such that \( \lim_{n \to \infty} \Delta^j x_n = 0 \) for \( j = 0, 1, \ldots, p - 1 \) when \( (-1)^{p-1} = -1 \).

Furthermore, if (3) holds, then (14) is also a necessary condition for the above conclusion to hold.

PROOF. Assume for the sake of a contradiction that \( \{x_n\} \) is a bounded nonoscillatory solution of equation (10). We may assume that \( \{x_n\} \) is eventually positive, say \( x_n > 0 \) on \( N(n_1) \) for some \( n_1 \geq n_0 \). The case \( \{x_n\} \) being eventually negative is similar. Let \( x_n = y_n + \phi_n \). If \( y_n \leq 0 \), then we have \( 0 < x_n \leq \phi_n \). But this contradicts our assumption that \( \phi_n \) is oscillatory. Thus, \( \phi_n > 0 \) on \( N(n_1) \). Due to sign condition on \( f \), it follows from equation (10) that
\[
\delta \Delta^p y_n = -q_nf(x_{n-k}) < 0, \quad n \in N(n_1).
\] (15)
This means that \( \{\Delta^j y_n\}, \ j = 0, 1, 2, \ldots, p - 1 \) are monotone sequences. Since \( \{y_n\} \) is bounded, it follows from Lemmas 1 and 2 that \( l \in \{0, 1\} \) and for \( (-1)^{p-1} = 1 \),
\[
\Delta^j y_n > 0, \quad \text{for} \ j = 0, 1, \ldots, l,
\]
\[
(-1)^{j-l} \Delta^j y_n > 0, \quad \text{for} \ j = l + 1, \ldots, p - 1.
\] (16)
Multiplying (15) by \( n^{(p-1)} \) and summing from \( n_1 \) to \( n - 1 \), we obtain
\[
\sum_{s=n_1}^{n-1} s^{(p-1)} \delta \Delta^p y_s + \sum_{s=n_1}^{n-1} s^{(p-1)} q_s f(x_{s-k}) = 0.
\] (17)
Applying Lemma 3 to the first term on the left-hand side of (17), we have
\[
\sum_{s=n_1}^{n-1} s^{(p-1)} \delta \Delta^p y_s = \sum_{k=1}^{p-1} (-1)^{k+1} \delta \Delta^{k-1} s^{(p-1)} \frac{\Delta^{p-k-1} y_{s+k-1}}{s=n_1}
\]
\[
+ (-1)^{p+1} \delta(1-n)^{p-1}\frac{y_p}{s=n_1}
\]
\[
= \sum_{k=1}^{p-1} (-1)^{k+1} \delta \Delta^{k-1} n_1^{(p-1)} \Delta^{p-k} y_{n+k-1}
\]
\[
+ (-1)^{p+1} \delta(1-n)[y_{n+p-1} - y_{n+p-1}] - K,
\]
where in view of (16),
\[
K = \sum_{k=1}^{p-1} (-1)^{k+1} \delta \Delta^{k-1} n_1^{(p-1)} \Delta^{p-k} y_{n,k-1} > 0.
\]
Therefore, we obtain
\[
\sum_{s=n_1}^{n-1} s^{(p-1)} q_s f(x_{s-k}) \leq K + (-1)^{p} \delta(1-n)[y_{n+p-1} - y_{n+p-1}] - K
\] (18)
Since \( \{y_n\} \) is bounded and (2) holds, it follows from (18) that
\[
\lim_{n \to \infty} f(x_{n-k}) = 0.
\]
This implies that
\[ \lim \inf_{n \to \infty} x_n = 0. \]
On the other hand, the bounded sequence \( \{y_n\} \) being positive and monotone has a finite limit
\( L \geq 0 \). Since \( \lim_{n \to \infty} \phi_n = 0 \) and \( y_n = x_n - \phi_n \), it follows that the limit \( L \) must be zero.

Clearly \( L = 0 \) is not possible if \( l = 1 \) or equivalently if \((-1)^p = 1\). Suppose that \( l = 0 \).
Then, \((-1)^p = -1\) and since \( x_n = y_n + \phi_n \) and \( \lim_{n \to \infty} \Delta^j \phi_n = \lim_{n \to \infty} \Delta^j y_n = 0 \) for
\( j = 0,1,2,\ldots,p-1 \), it follows that \( \lim_{n \to \infty} \Delta^j x_n = 0 \) for \( j = 0,1,2,\ldots,p-1 \).

Suppose now that \((3)\) holds, but \((14)\) is not satisfied. Clearly, by Theorem 1, equation \((10)\) has a bounded positive solution. This completes the proof.

**Theorem 5.** Let \( c > 1 \) and let \( \phi : \mathbb{N} \to \mathbb{R} \) be an oscillatory function such that \( \Delta^p \phi_n = h_n \) and \( \lim_{n \to \infty} \phi_n = 0 \). If \((14)\) is satisfied, then every bounded solution \( \{x_n\} \) of equation \((10)\) is oscillatory when \((-1)^p = -1\), and is either oscillatory or such that \( \lim_{n \to \infty} x_n = 0 \) when \((-1)^p = 1\).

Furthermore, if \((3)\) holds and \((7)\) is satisfied with \([a,b] = [c,2c]\), then \((14)\) is also a necessary condition for the above conclusion to hold.

**Proof.** Suppose that equation \((10)\) has a nonoscillatory solution \( \{x_n\} \). We may assume that \( \{x_n\} \) is eventually positive. Set
\[ z_n = x_n - cx_{n-k} \]
and
\[ y_n = z_n - \phi_n. \]
We claim that \( z_n \) is eventually negative; otherwise eventually
\[ x_n > cx_{n-l} \]
and by induction
\[ x_{n+ml} > c^m x_n. \]
Since \( x_n \) is bounded, this last inequality leads to a contradiction.

We next claim that \( y_n \) is eventually negative. For if \( y_n > 0 \) eventually, then,
\[ -\phi_n > -z_n > 0, \]
which, however, contradicts to our assumption that \( \phi_n \) is oscillatory.

Now by equation \((10)\),
\[ \delta \Delta^p y_n = -q_n f(x_{n-k}) < 0. \]
Since \( y_n < 0 \) and \( \delta \Delta^p y_n < 0 \), applying Lemmas 1 and 2, it follows that \( l \in \{0,1\} \), and for \( n \in N(n_1) \) and \((-1)^{p-l} = 1\),
\[ \Delta^j y_n < 0, \quad \text{for } j = 0,1,\ldots,l, \]
\[ (-1)^{j-l} \Delta^j y_n < 0, \quad \text{for } j = l+1,\ldots,p-1. \]
(19)

As in the proof of Theorem 1, we obtain the following inequality:
\[ \sum_{s=n_1}^{n-1} g^{(p-1)} q_s f(x_{s-k}) \leq K + (-1)^p \delta(p-1)! [y_{n+p-1} - y_{n_1+p-1}], \]
(20)
where due to \((19)\),
\[ K = \sum_{k=1}^{p-1} (-1)^{k+1} \delta k^{(p-1)} n_1^{k-1} \Delta^{p-k} y_{n_1+k-1} > 0. \]
It follows from (14) and (20) that
\[
\liminf_{t \to \infty} x_n = 0. \tag{21}
\]
On the other hand, \(\lim_{n \to \infty} y_n = L \leq 0\) exists, and also \(\lim_{n \to \infty} z_n = L\). Suppose that \(L < 0\).

Clearly, there exists a number \(n_2 \in N(n_0)\) such that for \(n \in N(n_2)\),
\[
2L < x_n - cx_{n-k} < \frac{L}{2}.
\]
But then we have
\[
x_{n-k} > \frac{-L}{2c},
\]
for \(n \in N(n_2)\), a contradiction with (21). Since \(L < 0\) whenever \(l = 1\), we can conclude that every bounded solution is oscillatory if \((-1)^p \delta = -1\).

Suppose that \(L = 0\), which is possible only if \(l = 0\). Then, as in [12], one can easily show that \(\lim_{n \to \infty} x_n = 0\).

To complete the proof, we note that if (14) fails and in addition (3) and (7), \([a, b] = [c, 2c]\) hold, then by Theorem 2, equation (10) has a bounded positive solution. This completes the proof.

EXAMPLE 2. Consider the difference equation
\[
\Delta^3(x_n - 2x_{n-3}) + 4z^3_{n-10} = 20(-1)^{n-1}.
\]
It is easy to check that the conditions of Theorem 5 are satisfied, and hence, every bounded solution of this equation is oscillatory. In fact, \(x_n = (-1)^n\) is such a solution.

REFERENCES