Controllability of two-point nonlinear boundary-value problems by the numerical-analytic method

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Abstract

By employing a numerical-analytic method, we establish sufficient conditions for the controllability of systems

\[ \begin{align*}
\frac{dy}{dt} &= A(t)y + B(t)u + g(t) + f(t,y,z,u) \\
\frac{dz}{dt} &= F(t,y,z,u),
\end{align*} \]

with boundary conditions of the form

\[ y(0) = a, \quad y(T) = c, \quad z(0) = b, \quad z(T) = d \]

or

\[ y(0) = a, \quad y(T) = c, \quad Q_0z(0) + Q_1z(T) = b, \]

where all functions involved are continuous in their domain of definition, \( T \) is a fixed real number, \( Q_j, j = 0, 1, \) are constant matrices, \( y \in \mathbb{R}^n, z \in \mathbb{R}^k, \) and \( u \in \mathbb{R}^m. \)

Keywords: Two point boundary-value problem; Numerical-analytic method; Nonlinear controllability solvability

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1. Introduction

In this paper we are concerned with the controllability of nonlinear differential equations of the form

\[
\begin{align*}
\frac{dy}{dt} &= A(t)y + B(t)u + g(t) + f(t, y, z, u) \\
\frac{dz}{dt} &= F(t, y, z, u),
\end{align*}
\] (1.1)

subject to boundary conditions

\[
y(0) = a, \quad y(T) = c, \quad z(0) = b, \quad z(T) = d
\] (1.2)

or

\[
y(0) = a, \quad y(T) = c, \quad Q_0z(0) + Q_1z(T) = b,
\] (1.3)

where \( T \) is a fixed real number, \( Q_j, j = 0, 1 \), are constant matrices, \( y \in \mathbb{R}^n \), \( z \in \mathbb{R}^k \), and \( u \in \mathbb{R}^m \). It is assumed that all functions involved are continuous in their domain of definition.

The problems (1.1) and (1.2) is to be referred to as problem \( \gamma_1 \), and (1.1) and (1.3) as problem \( \gamma_2 \). As usual, we say that \( \gamma_i \) is solvable if there is a control function \( u \) for which the problem admits a solution.

The controllability of boundary-value problems were investigated by many authors, see for instance [1–8] and the references cited therein. A summary and discussion of the methods employed for studying the controllability of nonlinear systems can be found in [1].

To the best of our knowledge the controllability of systems of the form (1.1) have been rarely considered in the literature due to a difficulty caused by the nonlinearity of the function \( F \). In fact, even if \( F \) can be linearized, the combined system might only be investigated under some conditions not generally fulfilled. For instance, condition (2.1) may not be satisfied for the whole system. The advantage of our method lies in the fact that we need only one of the equations to be quasilinear, which surely suffices to construct a control function \( u \) necessary to deal with the problem. Our technique is based on a numerical-analytic method introduced in [9] and on their derivatives used in [10–13]. It is worth mentioning that the numerical-analytic method have certain similarities with many of the other methods available for investigating boundary-value problems, see for example [14–18]. It is therefore natural expect that the method can be employed to investigate a wide range of boundary-value problems of different nature such as the one in this work.
2. Preliminaries

Let $G$ be a set defined by

$$G = \{(t, y, z, u) | 0 \leq t \leq T, \ (y, z, u) \in G_y \times G_z \times G_u\},$$

where $G_y \times G_z \times G_u$ is a compact subset of $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^m$. If $f$ is a function defined on $G$, then by $\|f\|_0$ we shall mean $\|f\|_0 = \max_G \|f\|$, where $\|f\|$ denotes the Euclidean norm of $f$.

Let $Y(t), Y(0) = I$, be the matrix solution of $(dy/\!dt) = A(t)y$, and $S(t)$ and $V(t)$ be matrices defined by

$$S(t) = Y^{-1}(t)B(t)$$

and

$$V(t) = \int_0^t S(\tau)S^T(\tau) \, d\tau.$$

Assuming that

$$\det V(T) \neq 0$$

(2.1)

we set

$$M_1 = \|f\|_0,$$

$$M_2 = \|F\|_0,$$

$$M_3 = \max\{M_{31}, M_{32}, M_{33}\},$$

where

$$M_{31} = \max_{t, \tau} \|S^T(t)V^{-1}(T)Y^{-1}(\tau)\|,$$

$$M_{32} = \max_{t, \tau} \|Y(t)Y^{-1}(\tau)\|,$$

$$M_{33} = \max_{t, \tau} \|Y(t)V(t)V^{-1}(T)Y^{-1}(\tau)\|.$$ 

We also assume that there exist positive real numbers $\ell_1$ and $\ell_2$ such that for all $(t, y_1, z_1, u_1)$ and $(t, y_2, z_2, u_2) \in G$,

$$\|f(t, y_1, z_1, u_1) - f(t, y_2, z_2, u_2)\| \leq \ell_1 (\|y_1 - y_0\| + \|z_1 - z_2\| + \|u_1 - u_2\|)$$

and

$$\|F(t, y_1, z_1, u_1) - F(t, y_2, z_2, u_2)\| \leq \ell_2 (\|y_1 - y_2\| + \|z_1 - z_2\| + \|u_1 - u_2\|).$$

Moreover, we require that

$$3\ell_1 M_3 + \ell_2 \left[ \frac{3\ell_1 M_3 T}{2} + \frac{\ell_2 T}{3} \right] < 1.$$ 

(2.2)
Finally, we give the following two lemmas which we rely on. Although these lemmas can be found in [9] we give their proofs here for convenience. The lemmas are concerned with the function $a(t)$ defined by

$$a(t) = 2t \left(1 - \frac{t}{T}\right), \quad t \in [0, T].$$

**Lemma 2.1.** If $u(t)$ is a continuous function defined on $[0, T]$, then

$$\left\| \int_0^t \left[ \varphi(s) - \frac{1}{T} \int_0^T \varphi(\xi) d\xi \right] ds \right\| \leq a(t) \| \varphi \|_0$$

for all $t \in [0, T]$.

**Proof.** Since

$$\int_0^t \left[ \varphi(s) - \frac{1}{T} \int_0^T \varphi(\xi) d\xi \right] ds = \left(1 - \frac{t}{T}\right) \int_0^t \varphi(s) ds - \frac{t}{T} \int_t^T \varphi(s) ds,$$

we see that

$$\left\| \int_0^t \left[ \varphi(s) - \frac{1}{T} \int_0^T \varphi(\xi) d\xi \right] ds \right\| \leq \left(1 - \frac{t}{T}\right) \int_0^t \| \varphi \|_0 ds + \frac{t}{T} \int_t^T \| \varphi \|_0 ds.$$

**Lemma 2.2.** It is true that

$$\left(1 - \frac{t}{T}\right) \int_0^t \varphi(s) ds + \frac{t}{T} \int_t^T \varphi(s) ds \leq \frac{T}{3} a(t)$$

for all $t \in [0, T]$.

**Proof.** Evaluating the integrals, we have

$$\left(1 - \frac{t}{T}\right) \int_0^t \varphi(s) ds + \frac{t}{T} \int_t^T \varphi(s) ds = a(t) \left[ \frac{T}{6} + \frac{a(t)}{3} \right].$$

Since $a(t) \leq \frac{T}{2}$ for all $t \in [0, T]$, the proof is complete. \qed

In what follows, we fix $u^0 \in L_2(0, T)$ such that

$$\int_0^T S(t)u^0(t) dt = 0. \quad (2.3)$$
3. Solvability of problem γ₁

In this section we investigate the solvability problem of solutions of (1.1) satisfying (1.2).

For our purpose we let

\[ u_0(t) = S^T(t)V^{-1}(T) \left[ Y^{-1}(T)c - a - \int_0^T Y^{-1}(t)g(t)\,dt \right] + u^0(t), \]

\[ y_0(t) = Y(t) \left[ a + \int_0^t S(\tau)u_0(\tau)\,d\tau + \int_0^t Y^{-1}(\tau)g(\tau)\,d\tau \right], \]

\[ z_0(t) = b + \frac{t}{T}(d - b) \]

and denote by \( G(b) \) the set of points \((t, y, z, u)\) such that \( t \in [0, T] \) and

\[ ||y - y_0(t)|| < 2M_3M_1T, \]

\[ ||z - z_0(t)|| < \frac{M_2T}{2}, \]

\[ ||u - u_0(t)|| < M_3M_1T. \]

Our first theorem asserts that if \( \gamma_1 \) is solvable, then under some reasonable conditions its solution can be written as a uniform limit of a certain sequence.

**Theorem 3.1.** Suppose that (2.1)–(2.3) hold that

\[ G^0_z = \{ z_0 \in G_z | G(z_0) \subset G \} \text{ is not empty.} \]  
(3.1)

If \( \gamma_1 \) is solvable and \( \psi(t) = (\varphi(t), w(t), v(t)) \) denotes its solution, then \( \psi(t) \) is the uniform limit of the sequence \( \{ \psi_n(t) \} = \{ (y_n(t), z_n(t), u_n(t)) \} \), where

\[ y_{n+1}(t) = y_0(t) + Y(t) \left[ \int_0^t Y^{-1}(\tau)f(\tau, \psi_n(\tau))\,d\tau - V(t)V^{-1}(T) \int_0^T Y^{-1}(\tau)f(\tau, \psi_n(\tau))\,d\tau \right], \]

\[ z_{n+1}(t) = z_0(t) + \int_0^t \left[ F(\tau, \psi_n(\tau)) - \frac{1}{T} \int_0^T F(s, \psi_n(s))\,ds \right]\,d\tau, \]

\[ u_{n+1}(t) = u_0(t) - S^T(t)V^{-1}(T) \int_0^T Y^{-1}(\tau)f(\tau, \psi_n(\tau))\,d\tau. \]  
(3.2)

**Proof.** We shall first show that the sequence \( \{ \psi_n(t) \} \) converges uniformly for \( t \in [0, T] \). It is easy to see directly that

\[ ||u_n(t) - u_0(t)|| \leq M_3M_1T \]

and
By using Lemma 2.1 we also have

$$\|y_n(t) - y_0(t)\| \leq 2M_3M_1T.$$  

Therefore the sequence \{\psi_n(t)\} belongs to \(G_y \times G_z \times G_u\) for \(t \in [0, T]\).

Define

$$\|\psi_n(t)\| = \|y_n(t)\| + \|z_n(t)\| + \|u_n(t)\|.$$  

In view of (3.2) and the definition of \(a(t)\), it follows that

$$\|\psi_1 - \psi_0\| \leq 3M_3M_1T + M_2a(t) = p_1a(t) + q_1 \leq p_1T/2 + q_1,$$

where \(p_1 = M_2\) and \(q_1 = 3M_1M_3T\). Similarly,

$$\|\psi_2(t) - \psi_1(t)\| \leq 3\ell_1M_3(p_1T/2 + q_1) + \ell_2q_1a(t) + \ell_2p_1T/3 = p_2a(t) + q_2,$$

where \(p_2 = \ell_2q_1\) and \(q_2 = (3\ell_1M_3T/2 + \ell_2T/3)p_1 + 3\ell_1M_3q_1\). In fact, by using mathematical induction we obtain that for \(n \geq 1\),

$$\|\psi_{n+1}(t) - \psi_n(t)\| \leq p_{n+1}a(t) + q_{n+1} \leq p_{n+1}T/2 + q_{n+1},$$  \(^{3.3}\)

where \(p_n\) and \(q_n\) satisfy the following equation

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = H \begin{bmatrix} p_n \\ q_n \end{bmatrix} \text{ for } n \geq 1$$

with

$$H = \begin{bmatrix} 0 & \ell_2 \\ \frac{3\ell_1M_3T}{2} + \frac{\ell_2T}{3} & 3\ell_1M_3 \end{bmatrix}$$  \(^{3.4}\)

In view of (2.2), we see that the eigenvalues \(\lambda_1\) and \(\lambda_2\) of the matrix \(H\) are inside the unit circle, and moreover, \(-1 < \lambda_2 < 0 < \lambda_1 < 1\), and \(|\lambda_2| < \lambda_1\). If we denote \(\lambda = \lambda_1\), then it follows from (3.4) that

$$\|(p_n, q_n)\| \leq \delta_0\lambda^{n-1}$$  \(^{3.5}\)

for some positive \(\delta_0\). We should note that \(\delta_0\) can be computed explicitly if desired.

In view of (3.5) we easily obtain from (3.3) that

$$\|\psi_{j+i}(t) - \psi_j\| \leq \sum_{k=0}^{i-1} \|\psi_{j+k+1} - \psi_{j+k}\| \leq \sum_{k=0}^{i-1} (p_{j+k+1}T/2 + q_{j+k+1})$$

$$\leq \delta_0(1 + T^2/4)^{1/2} \sum_{k=0}^{i-1} \lambda^{j+k},$$  \(^{3.6}\)
which surely implies that \( \{ \psi_n \} \) converges uniformly for \( t \in [0, T] \) to \( \psi_\infty = (y_\infty, z_\infty, u_\infty) \), say. By using (3.2) it is not difficult to verify that \( \psi_\infty \) satisfies the following system:

\[
y(t) = y_0(t) + Y(t) \left[ \int_0^t Y^{-1}(\tau)f(\tau, \rho(\tau))d\tau - V(t)V^{-1}(T) \int_0^T Y^{-1}(\tau)f(\tau, \rho(\tau))d\tau \right],
\]

\[
z(t) = z_0(t) + \int_0^t \left[ F(\tau, \rho(\tau)) - \frac{1}{T} \int_0^T F(s, \rho(s)) ds \right]d\tau,
\]

\[
u(t) = u_0(t) - S^T(t)V^{-1}(T) \int_0^T Y^{-1}(\tau)f(\tau, \rho(\tau))d\tau,
\]

where \( \rho = (y, z, u) \).

We claim that \( \psi_\infty \) also solves the BVP

\[
\frac{dy}{dt} = A(t)y + B(t)u + g(t) + f(t, \rho(t)),
\]

\[
\frac{dz}{dt} = F(t, \rho) + \frac{d - b}{T} - \frac{1}{T} \int_0^T F(t, \rho(t)) dt,
\]

\[
y(0) = a, \quad y(T) = c, \quad z(0) = b, \quad z(T) = d
\]

Indeed, by the Cauchy’s formula

\[
y(t) = Y(t)a + Y(t) \int_0^t Y^{-1}(\tau)[B(\tau)u(\tau) + g(\tau) + f(\tau, \rho(\tau))]d\tau.
\]

If we use \( y(T) = c \) then we obtain

\[
Y^{-1}(T)c - a = \int_0^T S(t)u(t) dt + \int_0^T Y^{-1}(t)[g(t) + f(t, \rho(t))] dt.
\]

We may express the control \( u \) as

\[
u(t) = S(t)e + u^0(t),
\]

where \( e \in \mathbb{R}^m \) is a constant vector. Substituting \( u(t) \) into (3.11) and solving for \( e \) results in

\[
e = V^{-1}(T) \left\{ Y^{-1}(T)c - a - \int_0^T Y^{-1}(t)g(t) dt - \int_0^T Y^{-1}(t)f(t, \rho(t)) dt \right\}.
\]

Therefore from (3.12) we obtain the last expression in (3.7). The first equation in (3.7) now easily follows from (3.10) by inserting \( u(t) \). It is not also difficult to see that the second equation in (3.7) is satisfied.
Consider the function \( w(t) = (u(t), w(t), v(t)) \) which, in view of the condition of theorem, is the solution of BVP (1.1), (1.2). Since

\[
w(T) = w(0) + \int_0^T F(\tau, \psi(\tau)) \, d\tau,
\]

we have

\[
d - b = \int_0^T F(\tau, \psi(\tau)) \, d\tau
\]

and consequently \( \psi(t) \) is a solution of (3.8).

Finally, it is not difficult to see that

\[
\|\psi_n - \psi\| \leq p_{n+1} z(t) + q_{n+1} \leq p_{n+1} T/2 + q_{n+1},
\]

and hence \( \lim_{n \to \infty} \psi_n = \psi \). This completes the proof of the theorem.

Now let us consider system (1.1) with boundary condition

\[
y(0) = a, \quad y(T) = c, \quad z(0) = z_0, \quad z(T) = d.
\]

In view of (3.2) one can easily construct a sequence

\[
\{\psi_n(t, z_0)\} = \{y_n(t, z_0), z_n(t, z_0), u_n(t, z_0)\}
\]

which converges a solution \( \psi(t, z_0) \) of problem (3.8), (3.14).

This observation allows us to define an \( \varepsilon \)-approximate solution of \( \gamma_1 \).

**Definition 3.1.** We shall say that \( \psi(t, z_0) \) is an \( \varepsilon \)-approximate solution of \( \gamma_1 \) if for a given \( \varepsilon > 0 \),

\[
d - z_0 = \int_0^T F(s, \psi(s, z_0)) \, ds,
\]

whenever

\[
\|z_0 - b\| < \varepsilon.
\]

In view of Theorem 3.1 we may easily formulate the following theorem on the existence of an \( \varepsilon \)-approximate solution of \( \gamma_1 \).

**Theorem 3.2.** Suppose that the conditions of Theorem 3.1 are all satisfied. Then there is an \( \varepsilon \)-approximate solution of problem \( \gamma_1 \) if and only if for some \( z_0 \in B(b, \varepsilon) \)

\[
\Delta(z_0) \equiv d - z_0 - \int_0^T F(s, \psi(s, z_0)) \, ds = 0,
\]

where \( \psi(s, z_0) \) is the limit of approximations defined in (3.2) with \( b = z_0 \).
The following lemma shows that the mapping $\Delta$ defined by (3.17) is continuous.

**Lemma 3.1.** Suppose that \((2.1)\) and \((2.3)\) hold. Then $\Delta(z_0)$ is continuous on $G_z^0$ as long as it exists.

**Proof.** Let

$$
\bar{M}_1 = \max \left\{ \| S^T(t)V^{-1}(T)Y^{-1}(T) \| \right\},
$$

$$
\bar{M}_2 = \max \left\{ \| Y(t)S(\tau)S^T(\tau)V^{-1}(T)Y^{-1}(T) \| \right\},
$$

$$
\bar{M} = \max \{ \bar{M}_1, \bar{M}_2 \},
$$

$$
d_0 = \bar{M}(1 + T) + 1.
$$

In view of Lemmas 2.1 and 2.2 it follows that

$$
\| \psi_0(t, z_1^0) - \psi_0(t, z_2^0) \| \leq d_0 \| z_1^0 - z_2^0 \| \quad \text{for} \quad z_1^0, z_2^0 \in G_z^0.
$$

It is not also difficult to show that for $n \geq 1$,

$$
\| \psi_n(t, z_1^0) - \psi_n(t, z_2^0) \| \leq (d_0 + q_n + \bar{p}_nT/2)\| z_1^0 - z_2^0 \|,
$$

(3.18)

where

$$
\begin{bmatrix}
\bar{p}_{n+1} \\
\bar{q}_{n+1}
\end{bmatrix}
= H
\begin{bmatrix}
\bar{p}_n \\
\bar{q}_n
\end{bmatrix}
+ \begin{bmatrix}
\bar{p}_1 \\
\bar{q}_1
\end{bmatrix}
\quad \text{for} \quad n \geq 1
$$

with $\bar{p}_1 = d_0\ell_2$, $\bar{q}_1 = 3\bar{M}T\ell_1d_0$, and $H$ as in (3.4).

On the other hand, it follows as in (3.5) that

$$
\| (\bar{p}_{n+1}, \bar{q}_{n+1}) \| \leq \delta_0 \| (\bar{p}_1, \bar{q}_1) \| \sum_{i=1}^{n} \lambda^i
$$

Using this inequality, we see from (3.18) that

$$
\| \psi_n(t, z_1^0) - \psi_n(t, z_2^0) \| \leq \left[ d_0 + \delta_0 \| (\bar{p}_1, \bar{q}_1) \| (1 + T^2/4)^{1/2} \sum_{i=1}^{n} \lambda^i \right] \| z_1^0 - z_2^0 \|
$$

and hence

$$
\| \psi_\infty(t, z_1^0) - \psi_\infty(t, z_2^0) \| \leq \delta_1 \| z_1^0 - z_2^0 \|,
$$

(3.19)

where

$$
\delta_1 = d_0 + \frac{\delta_0 \| (\bar{p}_1, \bar{q}_1) \|}{1 - \lambda} (1 + T^2/4)^{1/2}.
$$
In view of (3.17) and (3.19) we observe that
\[ \|A(z_0^1) - A(z_0^2)\| \leq (1 + \ell_2 T \delta_1) \|z_0^1 - z_0^2\|. \]

This means that \( A \) is continuous. \( \square \)

Note that since the expression in (3.17) involves a limiting process it is impossible to find a zero of \( A(z_0) \). To overcome this difficulty we will make use of a sequence of functions \( \{A_j(z_0)\} \), defined by
\[ A_j(z_0) = \int_0^T F(t, \psi_j(t, z_0)) \, dt + d - z_0. \] (3.20)

The following definitions and lemmas, which are extracted from [19], are necessary for our purpose.

**Definition 3.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^k \) with boundary \( \partial \Omega \). Let \( P_x \) be a continuous vector field defined on \( \partial \Omega \) without zeros. The degree of the continuous mapping \( k P_x k/C0 1 \) \( P_x \) from \( \partial \Omega \) onto the unit sphere \( k x k = 1 \) in \( \mathbb{R}^k \), denoted by \( \gamma(P; \partial \Omega) \), is the rotation of the field \( P \) on \( \partial \Omega \).

**Lemma 3.2.** Let the continuous vector field \( P \) be defined on a closed domain \( \Omega \cup \partial \Omega \). If \( \gamma(P; \partial \Omega) \neq 0 \), then the field \( P \) vanishes for at least one point in the domain \( \Omega \).

**Definition 3.3.** Two vector fields \( P_0 \) and \( P_1 \) are said to be homotopic on \( \partial \Omega \) if there exists a continuous vector-function \( P(t; x) \) defined for \( 0 \leq t \leq 1 \) and \( x \in \partial \Omega \) with values in \( \mathbb{R}^k \) such that \( P(0; x) = P_0 x \), \( P(1; x) = P_1 x \), and that \( P(t; x) \neq 0 \), \( (t, x) \in [0, 1] \times \partial \Omega \).

**Lemma 3.3.** Homotopic vector fields have the same rotation.

**Definition 3.4.** A vector field \( P \), \( P_x = x - \Psi x \), defined on a Banach space \( E \) is said to be compact if \( \Psi \) is a compact operator in \( E \).

**Definition 3.5.** Let \( \Omega \) be a bounded domain in a Banach space \( E \). A compact vector field defined on \( \partial \Omega \) is said to be nondegenerate if it has no zeros in \( \partial \Omega \).

If \( \Psi \) is a compact operator defined in a neighborhood of an isolated fixed point \( x_0 \in E \), then it follows that the associated vector field \( P \) is nondegenerate on all spheres centered at \( x_0 \) and having sufficiently small radius \( r \). It turns out that the field has the same rotation on all these spheres. This common rotation is known as the index of the fixed point \( x_0 \) and is denoted by \( \gamma(x_0; \Psi) \).
Lemma 3.4. The index $\gamma(x_0; \Psi)$ is equal to the rotation of the field $P$ on the boundary of any bounded domain in which $x_0$ is the only fixed point of $\Psi$.

Theorem 3.3. In addition to the conditions of Theorem 3.1, suppose that for a given $\epsilon > 0$ there is a convex closed set $\Omega_{\epsilon} \subset B(b, \epsilon) \cap G_0^1$ such that for some $j \geq 1$ the map $A_j : \Omega_{\epsilon} \rightarrow \mathbb{R}^k$ has a unique singular point $z = z_j^0$ of nonzero index in $\Omega_{\epsilon}$, and

$$\inf_{z \in \partial \Omega_{\epsilon}} \| A_j(z) \| > \delta_0 \ell_2 (1 + T^2 / 4)^{1/2} \lambda'(1 - \lambda)^{-1}. \quad (3.21)$$

Then problem $\gamma_1$ has an $\epsilon$-approximate solution.

Proof. Let us first show that the vector fields $A_j$ and $A$ are homotopic on $\partial \Omega_{\epsilon}$. To see this we construct a vector function

$$V(s, z) = A_j(z) + s(A(z) - A_j(z))$$

defined for $(s, z) \in [0, 1] \times \partial \Omega_{\epsilon}$. In view of Lemma 3.1 we see that $V$ is continuous in $s$ and $z$. On the other hand, we see from (3.6) that

$$\|A(z) - A_j(z)\| \leq \frac{\ell_2}{T} \int_0^T \| \psi_\infty - \psi_j \| \, ds \leq \delta_0 \ell_2 (1 + T^2 / 4)^{1/2} \lambda'(1 - \lambda)^{-1} \quad (3.22)$$

and, hence, in view (3.21),

$$\|V(s, z)\| \geq \|A_j(z)\| - \|A_j(z) - A(z)\| > 0.$$ 

Thus $V$ does not vanish for any value of the parameter $s \in [0, 1]$.

Since the point $z_j^0 \in \Omega_{\epsilon}$ is the unique singular point of $A_j(z)$, by Lemma 3.4 we may deduce that the rotation of the vector fields $A_j$ on $\partial \Omega_{\epsilon}$ is not zero. It follows from Lemma 3.3 that the rotation of $A$ on $\partial \Omega_{\epsilon}$ is not zero as well, and so by using Lemma 3.2 we see that the equation $A(z) = 0$ has at least one solution $z = z^0$ in $\Omega_{\epsilon}$. This, in view of Theorem 3.2, completes the proof. 

Remark 3.1. It should be noted that the verification of the assumption that for a given $\epsilon > 0$ there is a convex closed set $\Omega_{\epsilon} \subset B(b, \epsilon) \cap G_0^1$ such that for some $j \geq 1$ the map $A_j : \Omega_{\epsilon} \rightarrow \mathbb{R}^k$ has a unique singular point $z = z_j^0$ of nonzero index in $\Omega_{\epsilon}$, requires defining an index of an isolated singular point. In case $k = 2$ this is no problem at all, see [19]. The case $k > 2$ is not so easy to handle. However, it is well known that if $A_j$ is a topological map of a neighborhood of the singular point then the index is either $-1$ or $+1$, see [20], and if $A_j$ is continuously differentiable with a nonzero Jacobian at $z_j^0$ then the index of $z_j^0$ cannot be zero.

Theorem 3.4. In addition to the conditions of Theorem 3.1, suppose that for a given sequence $\{ \epsilon_n \}$ such that $\epsilon_n > 0$ and $\lim_{n \rightarrow \infty} \epsilon_n = 0$, there exists a sequence $\{ \Omega_n \}$ of closed and convex sets such that $\Omega_n \subset B(b, \epsilon_n) \cap G_0^1$ and for some $j_n \geq 1$ the map $A_{j_n} : \Omega_n \rightarrow \mathbb{R}^k$ has a unique singular point $z = z_n$ of nonzero index in $\Omega_n$, and
Then, problem $\gamma_1$ is solvable.

Proof. By Theorem 3.3 there is a sequence $\{z_n^0\}, n \geq 1$, such that $A(z_n^0) = 0$. Letting $\epsilon_n \to 0$ we see that $z_n^0 \to b$ as $n \to \infty$. In view of Lemma 3.1 we may conclude that $A(b) = 0$. $\square$

Theorem 3.5. If problem $\gamma_1$ is solvable, then

$$\|A_j(z)\| \leq \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^j (1 - \lambda)^{-1} + (1 + \ell_2 \delta_1)\|z - b\|$$

for every $j \geq 1$ and $z \in G^0_j$.

Proof. By using (3.21), (3.22), and $A(b) = 0$, we see that

$$\|A_j(z)\| = \|A(z) + (A_j(z) - A(z))\| \leq \|A_j(z) - A(z)\| + \|A(z) - A(b)\|$$

$$\leq \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^j (1 - \lambda)^{-1} + (1 + \ell_2 \delta_1)\|z - b\|$$

for every $j \geq 1$ and $z \in G^0_j$. This completes the proof of the theorem. $\square$

The following contrapositive form of Theorem 3.5 is obvious.

Corollary 3.1. If there exist $j \geq 1$ and $z \in G^0_j$ such that

$$\|A_j(z)\| > \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^j (1 - \lambda)^{-1} + (1 + \ell_2 \delta_1)\|z - b\|$$

then problem $\gamma_1$ is not solvable.

Remark 3.2. It is possible to define an $\epsilon$-approximate solution of $\gamma_1$ with respect to $z(T) = d$ as was done with respect to $z(0) = b$. In that case, one can easily prove theorems similar to Theorems 3.2–3.5.

4. Solvability of problem $\gamma_2$

In the previous section the boundary conditions (1.2) did not allow us to obtain the exact solution of problem $\gamma_1$. To overcome this deficiency we consider a mixed type boundary condition (1.3) with respect to the variable $z$. The key is to introduce a parameterization with parameter $z_0 = z(0)$. In that case, we are able prove that problem $\gamma_2$ is solvable and the solution can be represented as a limit of a uniformly convergent sequence of functions.
We assume that
\[
\det(k_0Q_0 + k_1Q_1) \neq 0 \quad \text{for some real numbers } k_0 \text{ and } k_1,
\]
and let
\[
\begin{align*}
    u_0(t) &= S^T(t) V^{-1}(T) \left[ Y^{-1}(T)c - a - \int_0^T Y^{-1}(t) g(t) \, dt \right] + u^0(t), \\
    y_0(t) &= Y(t) \left[ a + \int_0^t S(\tau) u_0(\tau) \, d\tau + \int_0^t Y^{-1}(\tau) g(\tau) \, d\tau \right], \\
    z_0(t) &= z_0 + \left[ k_0 + \frac{t}{T} (k_1 - k_0) \right] [Q_0k_0 + Q_1k_1]^{-1} [b - (Q_0 + Q_1) z_0].
\end{align*}
\]

Now we define a sequence \( \{\psi_n\} = \{(y_n, z_n, u_n)\} \) as follows:
\[
\begin{align*}
    y_{n+1}(t) &= y_0(t) + Y(t) \left[ \int_0^t Y^{-1}(\tau) f(\tau, \psi_n(\tau)) \, d\tau \
                               - V(t) V^{-1}(T) \int_0^T Y^{-1}(\tau) f(\tau, \psi_n(\tau)) \, d\tau \right], \\
    z_{n+1}(t) &= z_0(t) + \int_0^t \left[ F(\tau, \psi_n(\tau)) - \frac{1}{T} \int_0^T F(s, \psi_n(s)) \, ds \right] \, d\tau, \\
    u_{n+1}(t) &= u_0(t) - S^T(t) V^{-1}(T) \int_0^T Y^{-1}(\tau) f(\tau, \psi_n(\tau)) \, d\tau.
\end{align*}
\]

It is easy to verify that the sequence \( \{\psi_n\} \) satisfies the boundary conditions (1.3). We denote by \( G(a, z_0) \) the set of points \((y, z, u)\) which satisfy for each fixed \( t \in [0, T]\) the following inequalities:
\[
\begin{align*}
    \|y - y_0(t)\| &\leq 2M_1 M_3, \\
    \|z - z_0 - k_0(k_0Q_0 + k_1Q_1)[b - (Q_0 + Q_1) z_0]\| &\leq \frac{M_2 T}{2} + \|(k_0 - k_1)(k_0Q_0 + k_1Q_1)^{-1} [b - (Q_0 + Q_1) z_0]\|, \\
    \|u - u_0(t)\| &\leq M_1 M_3 T.
\end{align*}
\]

**Theorem 4.1.** Suppose that (2.1)–(2.3), and (4.1) are satisfied, and that
\[
    G_y^0 \times G_z^0 = \{(a, z_0) \in G_y \times G_z | G(a, z_0) \subset G\}
\]

is not empty.

Then for any point \((a, z_0) \in G_y^0 \times G_z^0\) the sequence \( \{\psi_n\} \) converges uniformly to the function \( \psi_\infty \) satisfying...
can show that

\[ p_n \]

where

\[ h \]

Proof. Using Lemma 2.1 we see that

\[ \frac{dy}{dt} = A(t)y + B(t)u + g(t) + f(t, \psi(t)), \]

\[ \frac{dz}{dt} = F(t, \psi(t)) + \tilde{\Lambda}(z) \]

where

\[ \tilde{\Lambda}(z) = \frac{1}{T}(k_1 - k_0)(k_0Q_0 + k_1Q_1)^{-1}[b - (Q_0 + Q_1)z] - \frac{1}{T} \int_0^T F(s, \psi(s)) \, ds. \]

and (1.3), where \( \psi = (y, z, u) \).

Moreover, \( \psi_\infty \) is a solution of the system

\[ \frac{dy}{dt} = A(t)y + B(t)u + g(t) + f(t, \psi(t)), \]

\[ \frac{dz}{dt} = F(t, \psi(t)) + \tilde{\Lambda}(z) \]

where

\[ \tilde{\Lambda}(z) = \frac{1}{T}(k_1 - k_0)(k_0Q_0 + k_1Q_1)^{-1}[b - (Q_0 + Q_1)z] - \frac{1}{T} \int_0^T F(s, \psi(s)) \, ds. \]

Proof. Using Lemma 2.1 we see that

\[ \|u(t) - u_0(t)\| \leq M_1M_5T, \]

\[ \|y(t) - y_0(t)\| \leq 2M_2M_1T(T + 1), \]

\[ \|z(t) - z_0 - k_0(k_0Q_0 + k_1Q_1)^{-1}[b - (Q_0 + Q_1)z_0]\|
\leq \frac{M_2T}{2} + \|(k_0 - k_1)(k_0Q_0 + k_1Q_1)^{-1}[b - (Q_0 + Q_1)z_0]\|

and therefore \( \psi_n, t \in [0, T] \), is well defined.

It is also easy to see that

\[ \|\psi_1 - \psi_0\| \leq 3M_2M_1T + M_2x(t) \leq p_1x(t) + q_1 \leq p_1T/2 + q_1, \]

where \( p_1 = M_2, q_1 = 3M_1M_5T \). In fact, by using mathematical induction one can show that

\[ \|\psi_{n+1} - \psi_n\| \leq p_{n+1}x(t) + q_{n+1} \leq p_{n+1}T/2 + q_{n+1}, \quad n \geq 1, \]

where \( p_{n+1} = \ell_2q_n \) and \( q_{n+1} = (3\ell_1M_3T/2 + \ell_2T/3)p_n + 3\ell_1M_3q_n \). The rest of the proof is similar to that of Theorem 3.1, and hence it is omitted. \( \square \)

Let \( \lambda \) be the positive eigenvalue of the matrix \( H \) in (3.4) and

\[ \tilde{\Lambda}_j(z) = \frac{1}{T}(k_1 - k_0)(k_0Q_0 + k_1Q_1)^{-1}[b - (Q_0 + Q_1)z] - \frac{1}{T} \int_0^T F(s, \psi_j(s)) \, ds. \]
Theorem 4.2. In addition to the conditions of Theorem 3.1, suppose that there is a convex closed domain $\hat{G}_2 \subseteq G^0_2$ such that for a fixed $j \in \mathbb{N}$ the map $\Lambda_j : \hat{G}_2 \to \mathbb{R}^k$ has a unique singular point $z = z^*_j$ in $\hat{G}_2$ with nonzero index, and that
\[
\inf_{z \in \partial G^0_2} \| \Lambda_j(z) \| > \lambda^{-j} (1 - \lambda)^{-1} \ell_2 (1 + T^2/4)^{1/2}.
\]
Then problem $\gamma^*_2$ is solvable in $G^0_2 \times G^0_2$.

Proof. The proof is similar to that of Theorem 3.3 and hence is omitted. $\square$

In this paper we have considered two nonlinear boundary-value problems, namely problem $\gamma^*_1$ and problem $\gamma^*_2$. The first problem contains separated boundary conditions and the second has a mixed type one. As we have indicted in Section 1, the system under investigation is not easy to handle because of the lack of a linear part. By using the numerical-analytic method we have been able to overcome this difficulty in the case when one of the equations has a linear part. In particular, we have proved that $\gamma^*_1$ has an $\epsilon$-approximate solution and $\gamma^*_2$ is solvable. In both cases, the solutions are expressible as uniform limits of certain sequences. It is worth mentioning that each sequence is defined by successive approximations and therefore one can obtain approximate solutions if desired. We note that this is not a trivial problem and therefore it should be addressed under a separate work.

References