Radar Dish

Armature controlled dc motor

\[ \theta \]

Outside

\[ \theta_{D} \text{ output} \]

Inside

\[ \theta \text{ input} \]

Gearbox

dc amplifier

Control Transformer

Control Transmitter

Prof. Dr. Y. Samim Ünlüsoy
COURSE OUTLINE

I. INTRODUCTION & BASIC CONCEPTS
II. MODELING DYNAMIC SYSTEMS
III. CONTROL SYSTEM COMPONENTS
IV. STABILITY
V. TRANSIENT RESPONSE
VI. STEADY STATE RESPONSE
VII. DISTURBANCE REJECTION
VIII. BASIC CONTROL ACTIONS & CONTROLLERS

IX. FREQUENCY RESPONSE ANALYSIS

X. SENSITIVITY ANALYSIS
XI. ROOT LOCUS ANALYSIS
In this chapter:

- A short introduction to the steady state response of control systems to sinusoidal inputs will be given.
- Frequency domain specifications for a control system will be examined.
- Bode plots and their construction using asymptotic approximations will be presented.
In frequency response analysis of control systems, the steady state response of the system to sinusoidal input is of interest.

The frequency response analyses are carried out in the frequency domain, rather than the time domain.

It is to be noted that, time domain properties of a control system can be predicted from its frequency domain characteristics.
For an LTI system the Laplace transforms of the input and output are related to each other by the transfer function, $T(s)$.

In the frequency response analysis, the system is excited by a sinusoidal input of fixed amplitude and varying frequency.
Let us subject a stable LTI system to a sinusoidal input of amplitude $R$ and frequency $\omega$ in time domain.

$$r(t) = R \sin(\omega t)$$

The steady state output of the system will be again a sinusoidal signal of the same frequency, but probably with a different amplitude and phase.

$$c(t) = C \sin(\omega t + \phi)$$
FREQUENCY RESPONSE - INTRODUCTION

Phase and gain relationships

Gain
Output
Input
Phase

0°  180°  360°  540°
To carry out the same process in the frequency domain for sinusoidal steady state analysis, one replaces the Laplace variable $s$ with

$$s = j\omega$$

in the input output relation

$$C(s) = T(s)R(s)$$

with the result

$$C(j\omega) = T(j\omega)R(j\omega)$$
The input, output, and the transfer function have now become complex and thus they can be represented by their magnitudes and phases.

- **Input**: \( R(j\omega) = |R(j\omega)| \angle R(j\omega) \)
- **Output**: \( C(j\omega) = |C(j\omega)| \angle C(j\omega) \)
- **Transfer Function**: \( T(j\omega) = |T(j\omega)| \angle T(j\omega) \)
With similar expressions for the input and the transfer function, the input output relation in the frequency domain consists of the magnitude and phase expressions:

\[ C(j\omega) = T(j\omega)R(j\omega) \]

\[ |C(j\omega)| = |T(j\omega)||R(j\omega)| \]

\[ \angle C(j\omega) = \angle T(j\omega) + \angle R(j\omega) \]
For the input and output described by

\[ r(t) = R \sin(\omega t) \]

\[ c(t) = C \sin(\omega t + \phi) \]

the amplitude and the phase of the output can now be written as

\[ C = R |T(j\omega)| \]

\[ \phi = \angle T(j\omega) \]
Consider the transfer function for the general closed loop system.

\[ T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \]

For the steady state behaviour, insert \( s=j\omega \).

\[ T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} \]

\( T(j\omega) \) is called the Frequency Response Function (FRF) or Sinusoidal Transfer Function.
The frequency response function can be written in terms of its magnitude and phase.

$$T(j\omega) = |T(j\omega)| \angle T(j\omega)$$

Since this function is complex, it can also be written in terms of its real and imaginary parts.

$$T(j\omega) = \text{Re}[T(j\omega)] + j\text{Im}[T(j\omega)]$$
Remember that for a complex number be expressed in its real and imaginary parts:

- The magnitude is given by:
  \[ |z| = \sqrt{(a+bj)(a-bj)} = \sqrt{a^2 + b^2} \]

- The phase is given by:
  \[ \angle z = \tan^{-1} \frac{b}{a} \]
The magnitude and phase of the frequency response function are given by:

\[
|T(j\omega)| = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)}
\]

\[
\angle T(j\omega) = \angle G(j\omega) - \angle [1 + G(j\omega)H(j\omega)]
\]

These are called the gain and phase characteristics.
For a system described by the differential equation

\[ \ddot{x} + 2\dot{x} = y(t) \]

determine the steady state response \( x_{ss}(t) \) for a pure sine wave input \( y(t) = 3\sin(0.5t) \)
The transfer function is given by
\[
\ddot{x} + 2\dot{x} = y(t)
\]

The transfer function is
\[
T(s) = \frac{X(s)}{Y(s)} = \frac{1}{s(s+2)}
\]

Insert \( s = j\omega \) to get:
\[
T(j\omega) = \frac{1}{j\omega(j\omega + 2)}
\]

For \( \omega = 0.5 \ [\text{rad/s}] \):
\[
T(0.5j) = \frac{1}{0.5j(0.5j+2)} = \frac{1}{-0.25 + j}
\]
Multiply and divide by the complex conjugate.

\[
T(0.5j) = \left( \frac{1}{-0.25+j} \right) \left( \frac{-0.25-j}{-0.25-j} \right) = \frac{-0.25-j}{1+0.0625}
\]

\[
T(0.5j) = -0.235 - 0.941j
\]

Determine the magnitude and the angle.

\[
|T(0.5j)| = \sqrt{(-0.235)^2 + (-0.941)^2} = 0.97
\]

\[
\angle T(0.5j) = \tan^{-1} \left( \frac{-0.941}{-0.235} \right) = -104^\circ
\]
The steady state response is then given by:

\[
x_{ss}(t) = 3 \times 0.97 \sin \left( 0.5t - 104^\circ \right)
= 2.91 \sin \left( 0.5t - 104^\circ \right)
\]
Express the transfer function (input : F, output : y) in terms of its magnitude and phase.

\[ m\ddot{y} + c\dot{y} + ky = F \]

\[ G(s) = \frac{1}{ms^2 + cs + k} \]
FREQUENCY RESPONSE – Example 2b

- Insert \( s = j\omega \) in the transfer function to obtain the frequency response function.

\[
G(s) = \frac{1}{ms^2 + cs + k}
\]

\[
T(j\omega) = \frac{1}{m(\omega j)^2 + c(\omega j) + k} = \frac{1}{(k - m\omega^2) + c\omega j}
\]

- Write the FRF in \( a + bj \) form.
FREQUENCY RESPONSE – Example 2c

- Multiply and divide the FRF expression with the complex conjugate of its denominator.

\[
T(j\omega) = \frac{1}{(k - m\omega^2) + c\omega j} \frac{(k - m\omega^2) - c\omega j}{(k - m\omega^2)^2 + (c\omega)^2}
\]

\[
T(j\omega) = \left[\frac{(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2}\right] + \left[\frac{-c\omega}{(k - m\omega^2)^2 + (c\omega)^2}\right] j
\]

\[
T(j\omega) = \text{Re}[T(j\omega)] + \text{Im}[T(j\omega)] j
\]
Obtain the magnitude and phase of the frequency response function.

\[ |z| = \sqrt{a^2 + b^2} \]

\[
|T(j\omega)| = \sqrt{\left(\frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2}\right)^2} = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}
\]

\[
\angle z = \tan^{-1}\frac{b}{a}
\]

\[
\angle T(j\omega) = \tan^{-1}\frac{-c\omega}{(k - m\omega^2)}
\]
The open loop transfer function of a control system is given as:

\[ G(s) = \frac{300(s + 100)}{s(s + 10)(s + 40)} \]

Determine an expression for the phase angle of \( G(jw) \) in terms of the angles of its basic factors. Calculate its value at a frequency of 28.3 rad/s.

Determine the expression for the magnitude of \( G(jw) \) in terms of the magnitudes of its basic factors. Find its value in dB at a frequency of 28.3 rad/s.
\[
G(s) = \frac{300(s + 100)}{s(s + 10)(s + 40)}
\]

\[
\angle G(j\omega) = \angle 300 + \angle G(j\omega + 100) - \angle G(j\omega) - \angle G(j\omega + 10) - \angle G(j\omega + 40)
\]

\[
= 0 + \tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{40}\right)
\]

\[
= 0^\circ + \tan^{-1}\left(\frac{\omega}{100}\right) - 90^\circ - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{40}\right)
\]

\[
\angle G(28.3j) = 0^\circ + \tan^{-1}\left(\frac{28.3}{100}\right) - 90^\circ - \tan^{-1}\left(\frac{28.3}{10}\right) - \tan^{-1}\left(\frac{28.3}{40}\right)
\]

\[
= 0^\circ + 15.8^\circ - 90^\circ - 70.5^\circ - 35.3^\circ = -180^\circ
\]
\[ G(s) = \frac{300(s+100)}{s(s+10)(s+40)} \]

**FREQUENCY RESPONSE – Example 3c**

<table>
<thead>
<tr>
<th>[ G(j\omega) ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{300</td>
</tr>
</tbody>
</table>

\[ = \frac{300\sqrt{\omega^2+100^2}}{\omega\sqrt{\omega^2+10^2} \sqrt{\omega^2+40^2}} \]

\[ |G(28.3j)| = \frac{300\sqrt{28.3^2+100^2}}{28.3\sqrt{28.3^2+10^2} \sqrt{28.3^2+40^2}} \]
\[ = \frac{(300)(103.9)}{(28.3)(30.0)(49.0)} = 0.749 \]
- Typical gain and phase characteristics of a closed loop system.

\[ |T(j\omega)| \]

\[ \angle T(j\omega) \]

Typical values:
- \( M_r \) worst case
- |0.707| dB
- \( \omega_r \)
- \( BW \)
Similar to transient response specifications in time domain, frequency response specifications are defined.

- Resonant peak, $M_r$,
- Resonant frequency, $\omega_r$,
- Bandwidth, $BW$,
- Cutoff Rate.
Resonant peak, $M_r$:

This is the maximum value of the transfer function magnitude $|T(j\omega)|$.

$M_r$ depends on the damping ratio $\zeta$ only and indicates the relative stability of a stable closed loop system.

A large $M_r$ results in a large overshoot of the step response.

As a rule of thumb, $M_r$ should be between 1.1 and 1.5.

$$M_r = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$$
**Resonant frequency,** $\omega_r$:

This is the frequency at which the resonant peak is obtained.

$$\omega_r = \omega_n \sqrt{1 - 2\xi^2}$$

Note that resonant frequency is different than both the undamped and damped natural frequencies!
**FREQUENCY DOMAIN SPECIFICATIONS**

- **Bandwidth, BW:**
  
  This is the frequency at which the magnitude of the frequency response function, \(|T(j\omega)|\), drops to 0.707 of its zero frequency value.

- **BW is directly proportional to \(\omega_n\) and gives an indication of the transient response characteristics of a control system. The larger the bandwidth is, the faster the system responds.**
FREQUENCY DOMAIN SPECIFICATIONS

- **Bandwidth, BW**: It is also an indicator of robustness and noise filtering characteristics of a control system.

\[ \omega_{BW} = \omega_n \sqrt{\left(1 - 2\xi^2\right) + \sqrt{4\xi^4 - 4\xi^2 + 2}} \]
FREQUENCY DOMAIN SPECIFICATIONS

- **Cut-off Rate**: This is the *slope* of the magnitude of the frequency response function, $|T(j\omega)|$, at higher (above resonant) frequencies.

- It indicates the ability of a system to distinguish signals from noise.

- Two systems having the same bandwidth can have different cutoff rates.
The Bode plot of a transfer function is a useful graphical tool for the analysis and design of linear control systems in the frequency domain.

The Bode plot has the advantages that:

- it can be sketched approximately using straightline segments without using a computer.

- relative stability characteristics are easily determined, and

- effects of adding controllers and their parameters are easily visualized.
**Bode plots for three systems**

- **3-dB frequency = 100 Hz**
  - **Gain**: 0 dB
  - **Phase**: 0°
  - **20 dB/div. 2 to 600 Hz**
  - **90°/div.
    - Well tuned, little peaking (100-Hz bandwidth)

- **Excessive peaking**
  - **Gain**: 0 dB
  - **Phase**: 0°
  - **20 dB/div. 2 to 600 Hz**
  - **90°/div.
    - Peaking, indicating marginal stability

- **3-dB frequency = 50 Hz**
  - **Gain**: 0 dB
  - **Phase**: 0°
  - **20 dB/div. 2 to 600 Hz**
  - **90°/div.
    - Stable, but sluggish (50-Hz bandwidth)
The Bode plot consists of two plots drawn on semi-logarithmic paper.

1. **Magnitude of the frequency response function** in decibels, i.e.,

\[
20 \log |T(j\omega)|
\]

on a linear scale versus frequency on a logarithmic scale.

2. **Phase** of the frequency response function on a linear scale versus frequency on a logarithmic scale.
Bode Plot

Bode Diagram

Magnitude (dB)

Phase (deg)

Frequency (rad/sec)
It is possible to construct the Bode plots of the open loop transfer functions, but the closed loop frequency response is not so easy to plot.

It is also possible, however, to obtain the closed loop frequency response from the open loop frequency response.

Thus, it is usual to draw the Bode plots of the open loop transfer functions. Then the closed loop frequency response can be evaluated from the open loop Bode plots.
It is possible to construct the Bode plots by adding the contributions of the basic factors of $T(j\omega)$ by graphical addition.

Consider the following general transfer function.

$$T(s) = \frac{K \prod_{p=1}^{P} (1 + T_ps)}{s^N \prod_{m=1}^{M} (1 + T_ms) \prod_{q=1}^{Q} \left(1 + 2\xi_q \frac{s}{\omega_{nq}} + \frac{s^2}{\omega_{nq}^2}\right)}$$
The logarithmic magnitude of $T(j\omega)$ can be obtained by summation of the logarithmic magnitudes of individual terms.

\[
\log |T(j\omega)| = \log K + \sum_{p=1}^{P} \log |1 + j\omega \tau_p| - \\
\sum_{m=1}^{M} \log |1 + j\omega \tau_m| - \sum_{q=1}^{Q} \log \left|1 + \frac{2\xi_q}{\omega_n q} j\omega + \left(\frac{j\omega}{\omega_n q}\right)^2\right|
\]
Similarly, the phase of $T(j\omega)$ can be obtained by simple summation of the phases of individual terms.

$$\phi = \angle T(j\omega) = \sum_{p} \tan^{-1}\omega_Tp - N(90^\circ) - \sum_{m} \tan^{-1}\omega_Tm - \sum_{q} \tan^{-1}\left(\frac{2\xi_q\omega_nq}{\omega_n^2 - \omega^2}\right)$$
Therefore, any transfer function can be constructed from the four basic factors:

1. **Gain**, \( K \) - a constant,

2. **Integral**, \( 1/j\omega \), or derivative factor, \( j\omega \) - pole or zero at the origin,

3. **First order factor** - simple lag, \( 1/(1+j\omega T) \), or lead \( 1+j\omega T \) (real pole or zero),

4. **Quadratic factor** - quadratic lag or lead.

\[
\frac{1}{\sqrt{[1 + 2\xi\left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2]}} \quad \text{or} \quad 1 + 2\xi\left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2
\]
Some useful definitions:

- The magnitude is normally specified in decibels [dB].

The value of M in decibels is given by:

\[ M[\text{dB}] = 20 \log M \]

- Frequency ranges may be expressed in terms of decades or octaves.

**Decade**: Frequency band from \( \omega \) to \( 10\omega \).

**Octave**: Frequency band from \( \omega \) to \( 2\omega \).
Gain Factor K.

- The gain factor multiplies the overall gain by a constant value for all frequencies.
- It has no effect on phase.

\[ G(s) = K \]
\[ G(j\omega) = K \]
\[ M = 20\log|G(j\omega)| = 20\log(K) \text{ [dB]} \]
\[ \phi = 0 \]

M : magnitude, \( \phi \) : phase.
Integral Factor $1/j\omega$ – pole at the origin.

- Magnitude is a straight line with a slope of -20 dB/decade becoming zero at $\omega = 1$ [rad/s].
- Phase is constant at -90° at all frequencies.

$$G(s) = \frac{1}{s}, \quad G(j\omega) = \frac{1}{j\omega} = -\frac{1}{\omega}j$$

$$M = 20\log|G(j\omega)| = 20\log\left(\frac{1}{\omega}\right) = -20\log\omega$$

$$\phi = -90^\circ$$
BODE PLOT

Double pole at the origin.

- Simply double the slope of the magnitude and the phase, i.e., -40 dB/decade becoming zero at $\omega = 1 \text{ [rad/s]}$ and -180° phase.

$$G(s) = \frac{1}{s^2}, \quad G(j\omega) = \frac{1}{(j\omega)^2} = -\frac{1}{\omega^2}$$

$$M = 20\log|G(j\omega)| = 20\log\left(\frac{1}{\omega^2}\right) = -40\log\omega$$

$$\phi = -180^\circ$$
Derivative Factor $j\omega$ – zero at the origin.

- Magnitude is a straight line with a slope of 20 dB/decade becoming zero at $\omega = 1$ [rad/s].
- Phase is constant at $90^\circ$ at all frequencies.

$$G(s) = s, \ G(j\omega) = \omega j$$

$$M = 20 \log |G(j\omega)| = 20 \log(\omega)$$

$$\phi = 90^\circ$$
Double zero at the origin.

- Simply double the slope of the magnitude and the phase, i.e., 40 dB/decade becoming zero at $\omega = 1 \text{ [rad/s]}$ and $180^\circ$ phase.

\[ G(s) = s^2, \quad G(j\omega) = -\omega^2 \]

\[ M = 20\log|G(j\omega)| = 40\log(\omega) \]

\[ \phi = 180^\circ \]
BODE PLOT – First Order Factor

Simple lag (Real pole) \( 1/(1+j\omega T) \).

\[
G(s) = \frac{1}{1 + Ts}
\]

\[
G(j\omega) = \frac{1}{1 + j\omega T} \cdot \frac{1 - j\omega T}{1 - j\omega T} = \frac{1}{1 + \omega^2 T^2} - \frac{\omega T}{1 + \omega^2 T^2} j
\]

\[
M = 20\log |G(j\omega)| = 20\log \left( \frac{1}{\sqrt{1 + \omega^2 T^2}} \right)
\]

\[
M = -20\log \sqrt{1 + \omega^2 T^2} \text{ [dB]}
\]

\[
\phi = \tan^{-1} (-\omega T) = -\tan^{-1} \omega T
\]
BODE PLOT – First Order Factor

Simple lag (Real Pole) \( \frac{1}{1+j\omega T} \).

\[
M = -20\log \sqrt{1 + \omega^2 T^2} \quad [\text{dB}]
\]

For \( \omega \ll \frac{1}{T} \)

\[ M \approx -20\log 1 = 0 \quad [\text{dB}] \]

For \( \omega \gg \frac{1}{T} \)

\[ M \approx -20\log \omega T \quad [\text{dB}] \]
It is clear that the actual magnitude curve can be approximated by two straight lines.

For \( \omega \ll \frac{1}{T} \)

\[ M \approx -20 \log 1 = 0 \, [\text{dB}] \]

For \( \omega \gg \frac{1}{T} \)

\[ M \approx -20 \log \omega T \, [\text{dB}] \]
\( \omega_c = \frac{1}{T} \) is called the **corner (break) frequency**. Maximum error between the linear approximation and the exact value will be at the corner frequency.

\[
M = -20 \log \sqrt{1 + \omega^2 T^2} \ [\text{dB}]
\]

\[
M \left( \omega = \frac{1}{T} \right) = -20 \log \sqrt{2} \approx -3 \ [\text{dB}]
\]
### BODE PLOT – First Order Factor

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$0.1\omega_c$</th>
<th>$0.5\omega_c$</th>
<th>$\omega_c$</th>
<th>$2\omega_c$</th>
<th>$10\omega_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Error [dB]</strong></td>
<td>0.04</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0.04</td>
</tr>
</tbody>
</table>

*M[dB]*

![Diagram showing Bode plot for a first order factor](image)
Transfer function $G(s)=\frac{1}{1+Ts}$ is a **low pass filter**.

At low frequencies the magnitude ratio is almost one, i.e., the output can follow the input.

For higher frequencies, however, the output cannot follow the input because a certain amount of time is required to build up output magnitude (time constant!).

Thus, the higher the corner frequency the faster the system response will be.
Simple lag \( 1/(1+j\omega T) \).

\[ \phi = \tan^{-1}(-\omega T) = -\tan^{-1} \omega T \]

- For \( \omega < < \frac{0.1}{T} \), \( \phi \approx 0[^\circ] \)
- For \( \omega >> \frac{10}{T} \), \( \phi \approx -90[^\circ] \)
It is clear that the actual phase curve can be approximated by three straight lines.

\[ \phi \approx 0^\circ \]

\[ \phi \approx -90^\circ \]

Linear variation in the range

\[ \frac{0.1}{T} \leq \omega \leq \frac{10}{T} \]

In this case corner frequencies are: \( 0.1/T \) and \( 10/T \)
Thus the maximum error of the linear approximation is 5.7°.
Simple lead (Real zero) $1 + j\omega T$.

$G(s) = 1 + Ts$

$G(j\omega) = 1 + \omega Tj$

$M = 20\log|G(j\omega)| = 20\log\left(\sqrt{1 + \omega^2 T^2}\right)$

$M = 20\log\sqrt{1 + \omega^2 T^2} \,[\text{dB}]$

$\phi = \tan^{-1}(\omega T) = \tan^{-1}\omega T$
BODE PLOT – First Order Factor

Simple lead (Real zero) $1 + j\omega T$.

$$M = 20\log \sqrt{1 + \omega^2 T^2} \text{ [dB]}$$

- **For** $\omega \ll \frac{1}{T}$
  $$M \approx 20\log 1 = 0 \text{ [dB]}$$
- **For** $\omega \gg \frac{1}{T}$
  $$M \approx 20\log \omega T \text{ [dB]}$$
It is clear that the actual magnitude curve can be approximated by two straight lines.

For $\omega \ll \frac{1}{T}$

$$M \approx 20 \log \omega T \text{ [dB]}$$

For $\omega \gg \frac{1}{T}$

$$M \approx 20 \log 1 = 0 \text{ [dB]}$$
BODE PLOT – First Order Factor

Simple lead \(1 + j\omega T\).

\[\phi = \tan^{-1}(\omega T)\]

- For \(\omega \ll \frac{0.1}{T}\), \(\phi \approx 0^\circ\)
- For \(\omega \gg \frac{10}{T}\), \(\phi \approx 90^\circ\)
It is clear that the actual phase curve can be approximated by three straight lines.

\[ \phi \approx 90 ^{\circ} \]

Linear variation in the range
\[ \frac{0.1}{T} \leq \omega \leq \frac{10}{T} \]
As overdamped systems can be replaced by two first order factors, only underdamped systems are of interest here.

\[
G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}
\]

A set of two complex conjugate poles.

\[
G(j\omega) = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + 1}
\]

\[
M = 20\log|G(j\omega)| = -20\log\sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2} \quad [\text{dB}]
\]
BODE PLOT – Quadratic Factors

\[
M = 20 \log |G(j \omega)| = -20 \log \sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2} + \left(2 \xi \frac{\omega}{\omega_n}\right)^2 \ [\text{dB}]
\]

**Low frequency asymptote, \( \omega << \omega_n \):**

\[
M \approx -20 \log (1) = 0 \ [\text{dB}]
\]

**High frequency asymptote, \( \omega >> \omega_n \):**

\[
M \approx -20 \log \left(\frac{\omega}{\omega_n}\right)^2 = -40 \log \left(\frac{\omega}{\omega_n}\right) \ [\text{dB}]
\]

Low and high frequency asymptotes intersect at \( \omega = \omega_n \), i.e. corner frequency is \( \omega_n \).
Therefore the actual magnitude curve can be approximated by two straight lines.
BODE PLOT – Quadratic Factors

\[ \phi = \angle G(j\omega) = -\tan^{-1} \left( \frac{2\xi \frac{\omega}{\omega_n}}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \right) \]

At low frequencies, \( \omega \to 0 \):
\[ \phi \approx 0 [^\circ] \]

At \( \omega = \omega_n \):
\[ \phi \approx -90 [^\circ] \]

At high frequencies, \( \omega \to \infty \):
\[ \phi \approx -180 [^\circ] \]
Thus, the actual phase curve can be approximated by three straight lines.

Corner frequencies are: $\omega_n/10$ and $10\omega_n$. 
It is observed that, the linear approximations for the magnitude and phase will give more accurate results for damping ratios closer to 1.0.

The peak magnitude is given by:

\[ M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \]

The resonant frequency:

\[ \omega_r = \omega_n\sqrt{1 - 2\xi^2} \]
For $\xi=0.707$ :

$M_r=1$ (or $M=20\log1=0$ dB).

Thus, there will be no peak on the magnitude plot.

- Note the difference that in transient response for step input, there will be no overshoot for critically or overdamped systems, i.e., for $\xi \geq 1.0$. 
BODE PLOT – Example 1a

- Sketch the Bode plots for the given open loop transfer function of a control system.

\[ T(s) = \frac{100000(1+s)}{s(s+10)(0.1s^2 + 14s + 1000)} \]

- First convert to standard form.

\[ T(j\omega) = \frac{100000(1+j\omega)}{(j\omega)(10)(1+0.1\omega j)(1000)} \left[ \left( j \frac{\omega}{100} \right)^2 + 1.4 \left( j \frac{\omega}{100} \right) + 1 \right] \]

\[ T(j\omega) = \frac{10(1+j\omega)}{(j\omega)(1+0.1\omega j) \left[ \left( j \frac{\omega}{100} \right)^2 + 1.4 \left( j \frac{\omega}{100} \right) + 1 \right] } \]
Identify the basic factors and corner frequencies:

- **Constant gain K**: \( K = 10, \ 20 \log_{10} 10 = 20 \ [\text{dB}] \)

- **First order factor (simple lead – real zero)**:
  \[ T = 1 \quad ( \omega_c = 1 / T = 1) - \text{for magnitude plot} \]

- **Integral factor**: \( 1 / j\omega \)

- **First order factor (simple lag – real pole)**:
  \[ T = 0.1 \quad ( \omega_c = 1 / T = 10) - \text{for magnitude plot} \]

- **Quadratic factor (complex conjugate poles)**:
  \( \omega_n = \omega_c = 100, \ \xi = 0.7 - \text{for magnitude plot} \)
Identify the basic factors and corner frequencies:

- **Constant gain K**: \( K = 10, \ 20 \log_{10} 10 = 20 \) [dB]
- **First order factor (simple lead – real zero)**: \( T = 1 \)
  \( (\omega_c^2 = 0.1/T = 0.1, \ \omega_c^3 = 10/T = 10) \) – for phase plot
- **Integral factor**: \( 1/j\omega \)
- **First order factor (simple lag – real pole)**: \( T = 0.1 \)
  \( (\omega_c^2 = 0.1/T = 1, \ \omega_c^3 = 10/T = 100) \) – for phase plot
- **Quadratic factor (complex conjugate poles)**: \( \omega_n = 100 \)
  \( (\omega_c^2 = \omega_n / 10 = 10, \ \omega_c^3 = 10\omega_n = 1000) \) – for phase plot
BODE PLOT – Example 1d

Bode (magnitude) plot

M[dB]

1 + jω

K = 10

ω [rad/s]

Quadratic factor

1/(1 + 0.1jω)

1/jω

Bode (magnitude) plot
BODE PLOT – Example 1e

Bode (phase) plot

\[ \frac{1}{1+0.1j\omega} \]

\[ \frac{1}{j\omega} \]

Quadratic factor

K=10
**BODE PLOT – Example 1f**

- **Matlab plot:** full blue lines
  - (just 4 lines to plot !)

```matlab
text = ['num=[100000 100000];
den=[0.1 15 1140 10000 0];
bode(num,den);
gr
'];
```

- **Approximate plots:** dashed lines
Transfer functions which have no poles or zeroes on the right hand side of the complex plane are called minimum phase transfer functions.

Nonminimum phase transfer functions, on the other hand, have zeros and/or poles on the right hand side of the complex plane.

The major disadvantage of Bode Plot is that stability of only minimum phase systems can be determined using Bode plot.
From the characteristic equation:

\[ 1 + G(s)H(s) = 0 \text{ or } G(s)H(s) = -1 \]

Then the magnitude and phase for the open loop transfer function become:

\[ 20 \log |G(j\omega)H(j\omega)| = 20 \log 1 = 0 \text{ dB} \]
\[ \angle G(j\omega)H(j\omega) = -180^\circ \]

Thus, when the magnitude and the phase angle of a transfer function are 0 dB and \(-180^\circ\), respectively, then the system is marginally stable.
If at the frequency, for which phase becomes equal to -180°, gain is below 0 dB, then the system is stable (unstable otherwise).

Further, if at the frequency, for which gain becomes equal to zero, phase is above -180°, then the system is stable (unstable otherwise).

Thus, **relative stability** of a minimum phase system can be determined according to these observations.
Gain Margin: Additional gain to make the system marginally stable at a frequency for which the phase of the open loop transfer function passes through -180°.

Phase Margin: Additional phase angle to make the system marginally stable at a frequency for which the magnitude of the open loop transfer function is 0 dB.
GAIN and PHASE MARGINS

Gain Margin

Phase Margin
Can you identify the transfer function approximately if the measured Bode diagram is available?