COURSE OUTLINE

I. INTRODUCTION & BASIC CONCEPTS
II. MODELING DYNAMIC SYSTEMS
III. CONTROL SYSTEM COMPONENTS
IV. STABILITY
V. TRANSIENT RESPONSE
VI. STEADY STATE RESPONSE
VII. DISTURBANCE REJECTION
VIII. BASIC CONTROL ACTIONS & CONTROLLERS
IX. FREQUENCY RESPONSE ANALYSIS
X. SENSITIVITY ANALYSIS
XI. ROOT LOCUS ANALYSIS
In this chapter:

- Open loop transfer functions of feedback control systems will be classified.
- Steady state error of feedback control systems due to step, ramp, and parabolic inputs will be investigated.
- Selection of controller parameters for a specified steady state error and the measures to be taken to reduce steady state error will be examined.
**Steady State Response** is the response of a system as time goes to infinity.

It is particularly important since it provides an indication of the accuracy of a control system when its output is compared with the desired input.

If they do not agree exactly, then a steady state error exists.
The Steady State Response of a system is judged by the steady state error due to step, ramp, and parabolic (acceleration) inputs.

These inputs may be associated with the ability of a control system to:

- Position itself relative to a stationary target,
- Follow a target moving at constant speed, and
- Track an object that is accelerating.
Note that for the steady state response to exist, the system must be stable.

Therefore before going into steady state analysis it would be good practise to check the stability of the system.
The steady state response and error can be obtained by using the final value theorem.

**The final value theorem:**

The final value of a time signal can be found from the Laplace transform of the signal in the s-domain.

\[
\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)
\]
The final value theorem:

In the case of the output of a system

\[ C(s) = G(s)R(s) \]

\[ \lim_{t \to \infty} c(t) = \lim_{s \to 0} sG(s)R(s) \]
The open loop transfer function of a general feedback control system is given by:

\[
\frac{B(s)}{R(s)} = G(s)H(s)
\]
The open loop transfer function of a **unity feedback** control system is given by:

\[
\frac{B(s)}{R(s)} = G(s)
\]
The Laplace transform of actuating error, \( E_a(s) \), for a general closed loop system:

\[
E_a(s) = R(s) - H(s)C(s) = R(s) - H(s)G(s)E_a(s)
\]

\[
E_a(s) = \frac{1}{1 + G(s)H(s)} R(s)
\]
Note that the *actuating* and *actual* errors will be identical only for a *unity feedback* system, i.e. with an ideal sensor.

For a non-unity feedback system, the actual error may not be zero when the actuating error is zero.
Here, with a systematic approach, it will be shown that the steady state error for a *unity feedback* system due to a certain type of input depends on the type of its *open loop* transfer function.

For a non-unity feedback system, the analysis is somewhat more involved* and will not be covered here.

* See Nise, Section 7.6, and Kuo, pp.248-249.
The open loop transfer function of a unity feedback control system can be written in the general form:

\[
T(s) = K \prod_{p=1}^{P} \left( 1 + T_p s \right)
\]

\[
s^N \prod_{m=1}^{M} \left( 1 + \tau_m s \right) \prod_{q=1}^{Q} \left( 1 + 2\xi_q \frac{s}{\omega_{nq}} + \frac{s^2}{\omega_{nq}^2} \right)
\]

A unity feedback control system with this open loop transfer function is called a type N system.

- \( s^N \): N poles at the origin (free integrators),
- \( K \): open loop gain.
N=1 : type 1 system

\[ G(s) = \frac{s^2}{(s+3)^2} + \frac{s+1}{s^3(s+2)} = \frac{s+1}{s(s+2)(s+3)^2} \]
SYSTEM TYPE – EXAMPLE 1b

Open loop gain : 1/18
Using the final value theorem:

\[ e_{ss} = \lim_{t \to \infty} e(t) \]

\[ = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} \]

Note that the steady state error depends on the input and the open loop transfer function of the system.
Step Input:

\[ r(t) = R \]

\[ R(s) = \frac{R}{s} \]

\[ e_{ss} = \lim_{s \to 0} \frac{sR}{1 + G(s)} \]

\[ = \frac{R}{1 + \lim_{s \to 0} G(s)} \]
Define the position error constant

\[ K_s = \lim_{s \to 0} G(s) \]

Then

\[ e_{ss} = \frac{R}{1 + K_s} \]
\[ K_s = G(0) \]
\[ e_{ss} = \frac{R}{1+K_s} \]

\[ G(s) = \frac{K(T_1s+1)(T_2s+1)...(T_ms+1)}{s^n(\tau_1s+1)(\tau_2s+1)...(\tau/ns+1)} \]

for type 0 systems

for type 1 or higher systems

for type 0 systems

for type 1 or higher systems
Ramp Input:

\[ r(t) = Rt \]

\[ L\{t^n\} = \frac{n!}{s^{n+1}} \]

\[ R(s) = \frac{R}{s^2} \]

\[ e_{ss} = \lim_{s \to 0} \frac{s\left(\frac{R}{s^2}\right)}{1 + G(s)} \]

\[ = \lim_{s \to 0} \frac{R}{s + sG(s)} \]

\[ = \lim_{s \to 0} \frac{R}{s} \cdot \lim_{s \to 0} sG(s) \]
Define the velocity error constant

\[ K_v = \lim_{s \to 0} sG(s) \]

Then

\[ e_{ss} = \frac{R}{K_v} \]
\[ K_v = \lim_{s \to 0} sG(s) \]

\[ e_{ss} = \frac{R}{K_v} \]

\[ G(s) = \frac{K(T_1 s + 1)(T_2 s + 1) \ldots (T_m s + 1)}{s^N(\tau_1 s + 1)(\tau_2 s + 1) \ldots (\tau_n s + 1)} \]

For type 0 systems:

- \( K_v = K \)
- \( e_{ss} = \frac{R}{K} \)

For type 1 systems:

- \( K_v = \infty \)
- \( e_{ss} = \frac{\infty}{0} \)

For type 2 or higher systems:

- \( K_v = \infty \)
- \( e_{ss} = \frac{R}{K} \)
Parabolic (acceleration) Input:

\[ r(t) = \frac{R}{2} t^2 \]

\[ L\{t^n\} = \frac{n!}{s^{n+1}} \]

\[ R(s) = \frac{R}{s^3} \]

\[ e_{ss} = \lim_{s \to 0} s \left( \frac{R}{s^3} \right) \frac{1 + G(s)}{s^2 + s^2 G(s)} \]

\[ = \lim_{s \to 0} \frac{R}{s^3} \frac{R}{s^2 + s^2 G(s)} \]

\[ = \lim_{s \to 0} s^2 G(s) \]
Define the acceleration error constant

\[ K_a = \lim_{s \to 0} s^2 G(s) \]

Then

\[ e_{ss} = \frac{R}{K_a} \]
\[ K_a = \lim_{s \to 0} s^2 G(s) \]

\[ e_{ss} = \frac{R}{K_a} \]

\[ G(s) = \frac{K(T_1 s + 1)(T_2 s + 1) \cdots (T_m s + 1)}{s^n(\tau_1 s + 1)(\tau_2 s + 1) \cdots (\tau_n s + 1)} \]

- For type 0 and 1 systems: 
  - \( K_a = K \)
  - \( e_{ss} = \frac{R}{K} \)
- For type 2 systems: 
  - \( K_a = \infty \)
  - \( e_{ss} = \infty \)
- For type 3 or higher systems: 
  - \( K_a = \infty \)
  - \( e_{ss} = \infty \)
<table>
<thead>
<tr>
<th>System Type</th>
<th>Step Input</th>
<th>Ramp Input</th>
<th>Parabolic Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{R}{1+K}$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$\frac{R}{K}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$\frac{R}{K}$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
For linear systems, the steady state error for the simultaneous application of two or more inputs will be the superposition of the steady state errors due to each input applied separately.
For example, for an input of 
\[ r(t) = 1 + 2t + \frac{3t^2}{2} \]
the steady state error is given as the superposition of the steady responses to each of the inputs.

\[ e_{ss} = \frac{1}{1+K_s} + \frac{2}{K_v} + \frac{3}{K_a} \]
Therefore the steady state error of the system subjected to the composite input will be:

\[
e_{ss} = \frac{1}{1 + K_s} + \frac{2}{K_v} + \frac{3}{K_a}
\]

For Type 0 systems:
\[
e_{ss} = \frac{1}{1 + K} + \frac{2}{0} + \frac{3}{0} = \infty
\]

For Type 1 systems:
\[
e_{ss} = \frac{1}{1 + \infty} + \frac{2}{K} + \frac{3}{0} = \infty
\]

For Type 2 systems:
\[
e_{ss} = \frac{1}{1 + \infty} + \frac{1}{\infty} + \frac{3}{K} = \frac{3}{K}
\]

For Type 3 or higher systems:
\[
e_{ss} = \frac{1}{1 + \infty} + \frac{1}{\infty} + \frac{3}{\infty} = 0
\]
It is observed that:

i) When $e_{ss}$ is finite, increasing the open loop gain decreases the steady state error.

Step: $\frac{R}{1+K}$

Ramp: $\frac{R}{K}$

Acceleration: $\frac{K}{K}$

K: open loop gain
ii) As the type of the system is increased, $e_{ss}$ decreases. Therefore one may attempt to improve the steady state response by including an integrator in the controller. This, however, may cause stability problems which become critical for type 3 or higher systems.
iii) The approach to the minimization of the steady state error in control systems may thus be:

Determine the maximum possible open loop gain and check the maximum allowable open loop gain that will not result in instability.

Choose the smaller of the two open loop gain values.
Determine the steady state error for \( r(t) = 2 + 3t \)

\[
G(s) = \frac{1000}{s(s+5)} = \frac{1000}{s(5)(0.2s+1)} = \frac{200}{s(0.2s+1)}
\]

\( N = 1 : \) type 1 system, \( K = 200 \)
STEADY STATE ERROR – Example 1b

N=1 : type 1 system, K=200

e_{ss} = 0 for r(t) = 2 (step input)

e_{ss} = \frac{R}{K} = \frac{3}{200} = 0.015 for r(t) = 3t (ramp input)

Hence e_{ss} = 0.015 = 1.5 \%

It is obvious that if you increase the value of the open loop gain K, say by increasing amplifier gain, steady state error will decrease.
Let us check the maximum value of the amplifier gain without causing instability.

\[
\frac{C(s)}{R(s)} = \frac{10K_a}{s(s+5)} = \frac{10K_a}{s^2 + 5s + 10K_a}
\]
Thus the characteristic polynomial is given by:

$$D(s) = s^2 + 5s + 10K_a$$

It is obvious that it passes Hurwitz test. Application of Routh’s stability criterion will result in only one condition on stability:

$$s^2 \quad 1 \quad 10K_a$$
$$s^1 \quad 5 \quad 0$$
$$s^0 \quad 10K_a$$

$$K_a > 0$$
As an alternative solution, let us calculate the steady state error using the final value theorem.

\[
C(s) = \frac{1000}{s(s+5)} + \frac{1000}{s(s+5) + 1000}
\]

\[
r(t) = 2 + 3t
\]

\[
R(s) = \frac{2}{s} + \frac{3}{s^2}
\]

\[
E(s) = R(s) - C(s) = R(s) - \frac{1000}{s(s+5) + 1000}R(s)
\]

\[
E(s) = \frac{s(s+5)}{s^2 + 5s + 1000}R(s) = \frac{s(s+5)}{s^2 + 5s + 1000}
\]

\[
\left(\frac{2}{s} + \frac{3}{s^2}\right)
\]
Thus with the error expression in Laplace domain:

\[
E(s) = \frac{s + 5}{s^2 + 5s + 1000} \left( \frac{2s + 3}{s} \right)
\]

\[
e_{ss} = \lim_{s \to 0} s E(s)
= \lim_{s \to 0} s \left( \frac{2s + 3}{s} \right) \frac{s + 5}{s^2 + 5s + 1000}
= \frac{15}{1000} = 0.015 = 1.5 \%
\]

It is obvious that this method requires considerably higher effort.
Determine the value of amplifier gain $K_a$ such that the steady state error for a ramp input $r(t)=3t$ is going to be at most 2%.

$$G(s) = \frac{300K_a}{s(s+2)(s+5)} = \frac{300K_a}{s(2)(5)(0.5s+1)(0.2s+1)} = \frac{30K_a}{s(0.5s+1)(0.2s+1)}$$

$N=1$: type 1 system, $K=30K_a$
\[ G(s) = \frac{30K_a}{s(0.5s+1)(0.2s+1)} \]

**N=1 : type 1 system, K=3K_a; R=3**

\[ e_{ss} = \frac{R}{K} = \frac{3}{30K_a} \leq 0.02 \text{ for } r(t) = 3t \text{ (ramp input)} \]

Hence \( K_a \geq \frac{3}{30(0.02)} = 5 \)

Now, the limit on \( K_a \) with respect to stability should be checked.
Let us check the maximum value of the amplifier gain without causing instability.

\[
\frac{C(s)}{R(s)} = \frac{\frac{30K_a}{s(s+2)(s+5)}}{1 + \frac{30K_a}{s(s+2)(s+5)}} = \frac{30K_a}{s^3 + 7s^2 + 10s + 30K_a}
\]
Thus the characteristic polynomial is given by:

\[ D(s) = s^3 + 7s^2 + 10s + 30K_a \]

It is obvious that it passes Hurwitz test. Application of Routh’s stability criterion will result in:

<table>
<thead>
<tr>
<th>(s^3)</th>
<th>1</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s^2)</td>
<td>7</td>
<td>30K_a</td>
</tr>
<tr>
<td>(s^1)</td>
<td>(\frac{70 - 30K_a}{7})</td>
<td>0</td>
</tr>
<tr>
<td>(s^0)</td>
<td>30K_a</td>
<td></td>
</tr>
</tbody>
</table>

\[ K_a > 0 \]

\[ 10 - \frac{30}{7} K_a > 0 \]

\[ K_a < \frac{70}{30} = 2.33 \]
It is obvious that the value of $K_a$ required for the specified steady state error will make the system unstable.

Therefore, the minimum possible steady state error will be higher than the desired value.

Can you determine the minimum possible steady state error for this system?