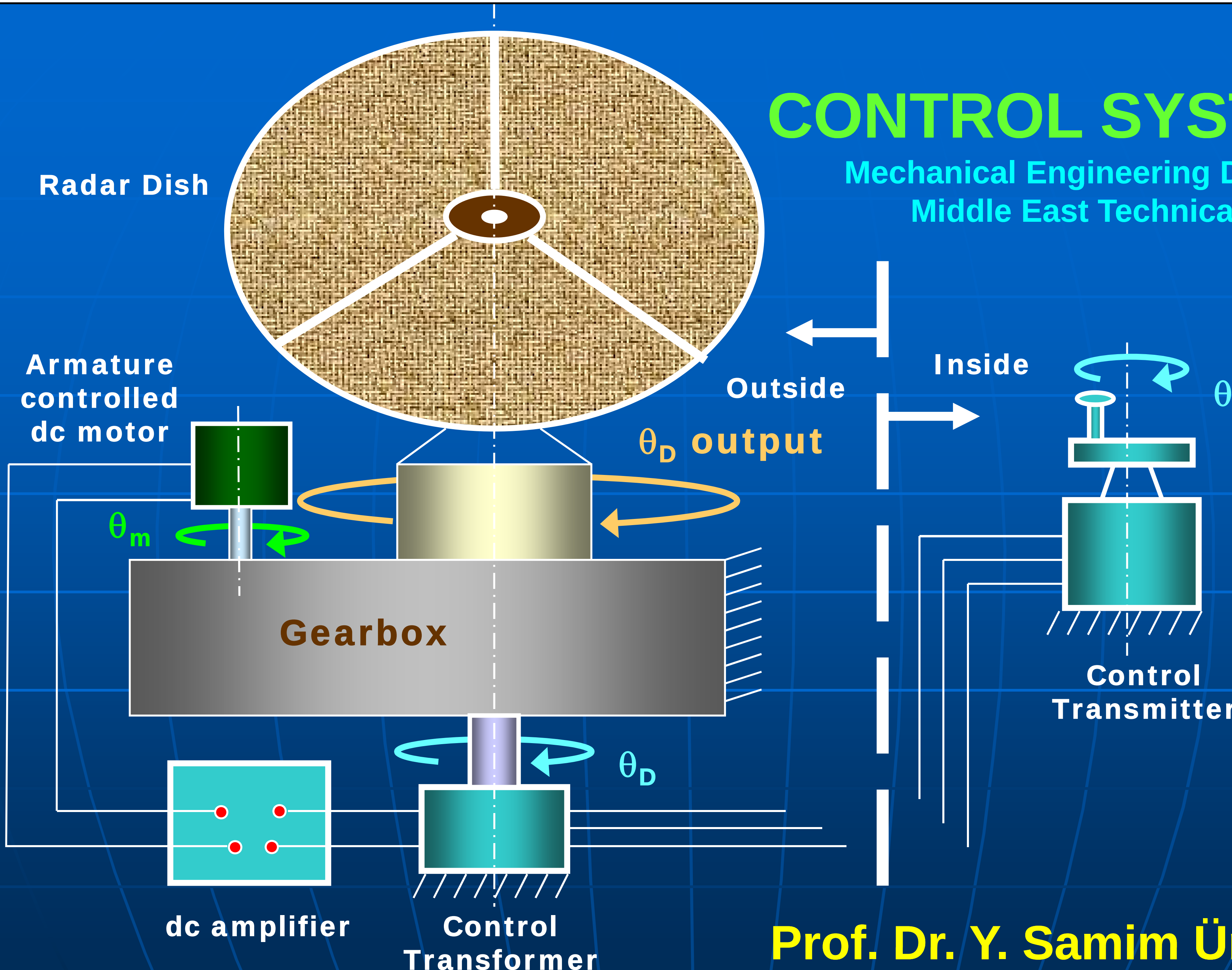
ME 304
CONTROL SYSTEMS

Mechanical Engineering Department,
Middle East Technical University



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COURSE OUTLINE

TRANSIENT RESPONSE OBJECTIVES

In this chapter :

- Time response of general first and second order systems to standard test inputs will be obtained.
- Specification of transient response as performance characteristics for control systems will be examined.
- The selection of controller parameters to meet transient response specifications will be explored.

CONTROL CRITERIA

- Certain requirements are to be met by a properly designed control system. These requirements involve, in general :
  - Satisfactory Transient response,
  - Stability,
  - Accuracy (Satisfactory steady-state response)
  - Disturbance rejection,
  - Minimum sensitivity to parameter variations or uncertainties in the plant and / or in the feedback path.

TIME RESPONSE

Dorf&Bishop Section 5.1

- The time response of a control system consists of two parts :
  - **Transient response** : the response from the initial state to the final state of the system.
  - **Steady-state response** : response of the system when time approaches infinity, i.e., at the final state.

STANDARD TEST INPUTS

- To obtain a basis for the evaluation and comparison of various control systems, the use of standard test inputs has been universally accepted.
- These simple and well defined inputs simplify analytical and experimental analyses of control systems.

STANDARD TEST INPUTS

Nise Table 1.1, Dorf&Bishop Sect. 5.2

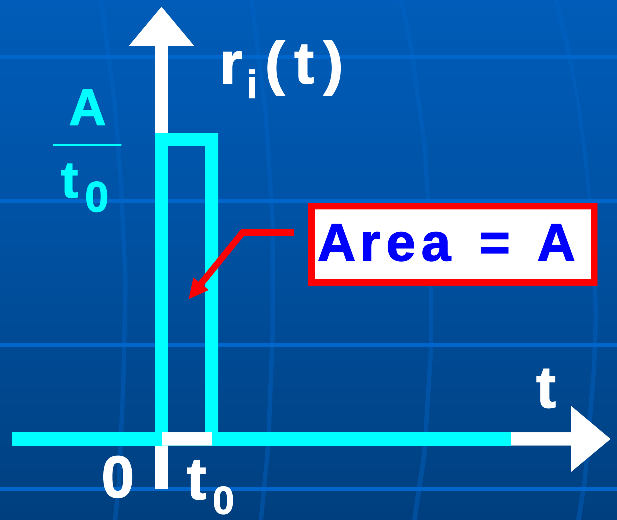
- Typical examples of standard input functions are :
  - Impulse function,
  - Step function,
  - Ramp function, and
  - Sinusoidal function.
- The response of first order systems to the first three inputs will now be examined.

STANDARD TEST INPUTS
Impulse Function

- The **impulse** function is defined as :

$$r_i(t) = \lim_{t_0 \to 0} \frac{A}{t_0} \qquad 0 < t < t_0$$

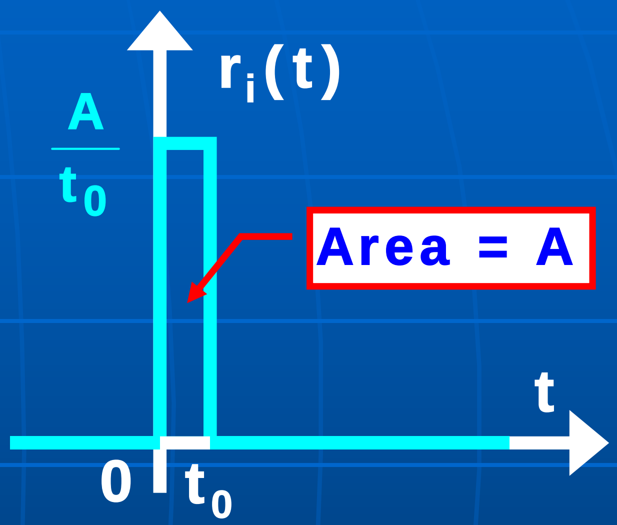
$$r_i(t) = 0 \qquad t < 0 \;\; \text{and} \;\; t > t_0$$



- Laplace Transform : $R_i(s) = A$

STANDARD TEST INPUTS
Impulse Function

- Used to represent inputs of very large magnitude and very short duration.
- Magnitude is measured by its area.



$A = 1 \;\Rightarrow\;$ Unit impulse : $R_{ui}(s) = 1$

STANDARD TEST INPUTS
Step Function

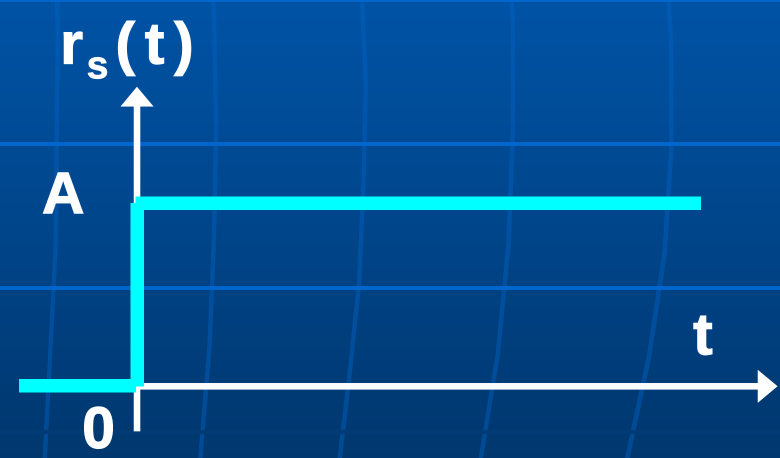
- The **step** function is defined as :

$$r_s(t) = A \qquad t \geq 0$$

k : constant

$$L\{k\} = \frac{k}{s}$$

$$R_s(s) = \frac{A}{s}$$



- Unit step Function : A=1 $\Rightarrow$ $R_{us}(s) = \frac{1}{s}$

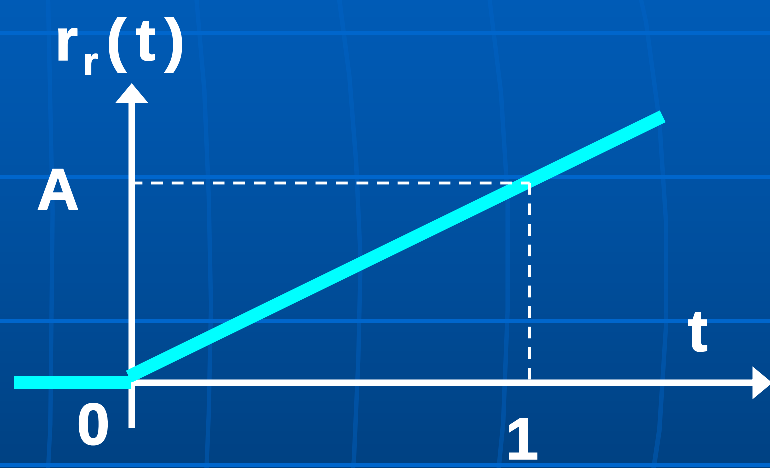
STANDARD TEST INPUTS
Ramp Function

- **The ramp function is defined as :**

$$r_r(t) = A\,t \qquad t \geq 0$$

$$L\{t\} = \frac{1}{s^2}$$

$$R_r(s) = \frac{A}{s^2}$$



- **Unit ramp function : A=1** $\Rightarrow$ $R_{ur}(s) = \dfrac{1}{s^2}$

STANDARD TEST INPUTS

- **Relation between standard test inputs :**
  - It may be observed that

$$r_i(t) = \frac{dr_s(t)}{dt} = \frac{d^2 r_r(t)}{dt^2}$$

  - Therefore, one can also write

$$c_i(t) = \frac{dc_s(t)}{dt} = \frac{d^2 c_r(t)}{dt^2}$$

  - Thus, if the response to one of the standard test inputs is available; the response to the other test inputs can be found simply by integration or differentiation.

STANDARD TEST INPUTS

- **Relation between responses to standard test inputs :**

$$c_i(t) = \frac{dc_s(t)}{dt}$$

$$c_s(t) = \int_0^t c_i(\tau)\,d\tau$$

$$c_s(t) = \frac{dc_r(t)}{dt}$$

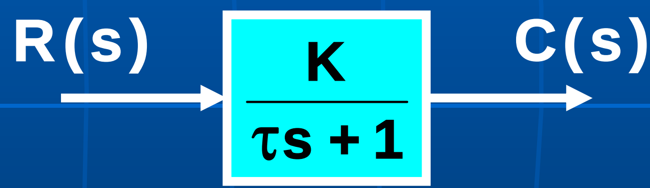
$$c_r(t) = \int_0^t c_s(\tau)\,d\tau$$

TRANSIENT RESPONSE of FIRST ORDER SYSTEMS

Nise Section 4.3

- **There exists a number of rather simple systems which may be represented by the same general transfer function :**

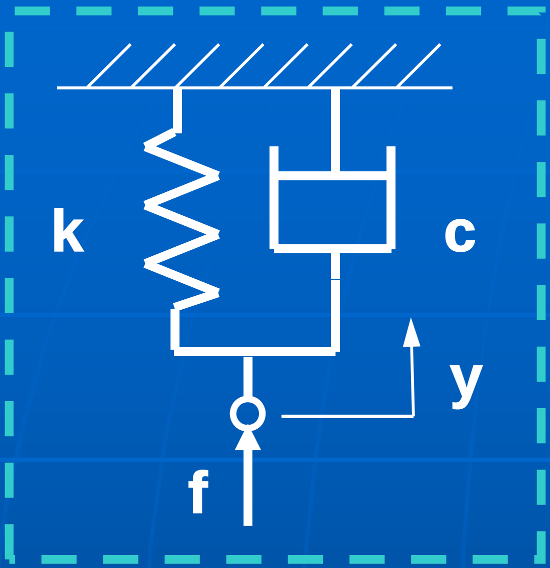
$$G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$$



K : gain.

τ : time constant.

FIRST ORDER SYSTEMS

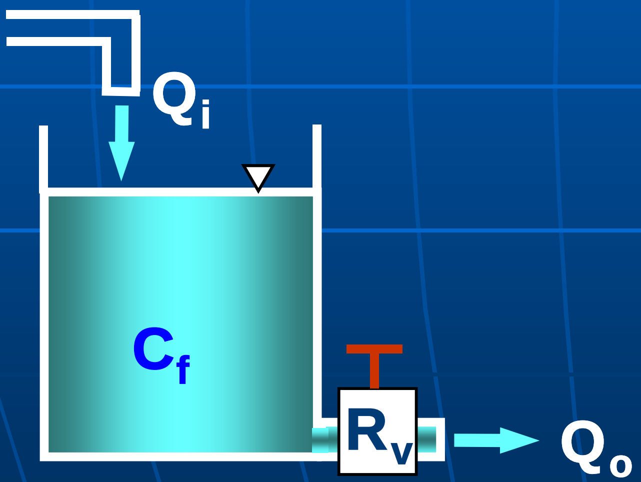


$$c\frac{dy}{dt}+ky=f$$

$$K=\frac{1}{k}$$

$$G(s)=\frac{\frac{1}{k}}{\frac{c}{k}s+1}$$

$$\tau=\frac{c}{k}$$



$$R_vC_f\frac{dQ_o}{dt}+Q_o=Q_i$$

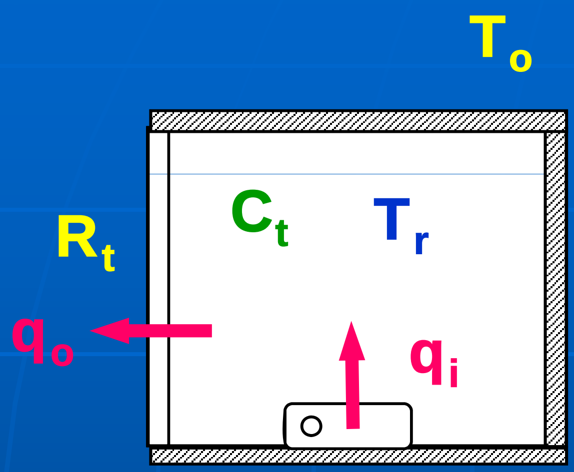
$$K=1$$

$$G(s)=\frac{1}{R_vC_fs+1}$$

$$\tau=R_vC_f$$

FIRST ORDER SYSTEMS

$\Delta T = T_r - T_o$

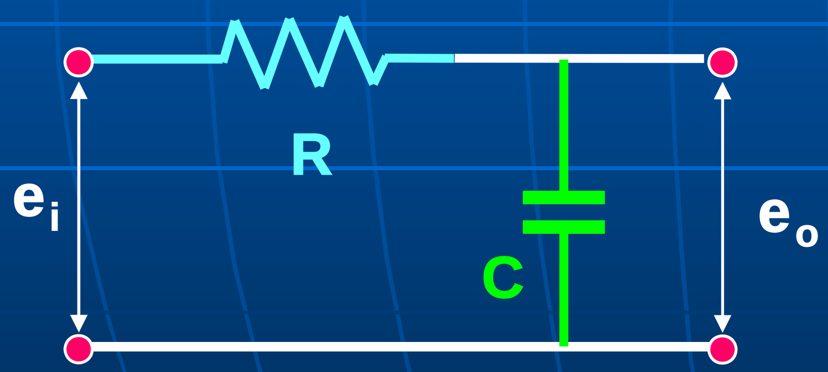


$$R_t C_t \frac{d\Delta T}{dt} + \Delta T = R_t q_i$$

$$K = R_t$$

$$G(s) = \frac{R_t}{R_t C_t s + 1}$$

$$\tau = R_t C_t$$



$$RC \frac{de_o}{dt} + e_o = e_i$$

$$K = 1$$

$$G(s) = \frac{1}{RCs + 1}$$

$$\tau = RC$$

FIRST ORDER SYSTEMS

- It is clear that different first order systems, irrespective of their actual physical construction can be represented by the same general transfer function.
- These system characteristics will be reflected in the general transfer functions by the definitions of the **gain** and **time** constants.
- K : Gain
- τ : Time constant

$$G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$$

$$R_i(s) = A$$

$$G(s) = \left(\frac{K}{\tau s + 1}\right)$$

FIRST ORDER SYSTEMS

(Ogata p. 222-223)

- **The impulse response (response to an impulse function) of the general first order system is :**

C(s)=G(s)R(s)

$$a = 1/\tau$$

$$C_i(s) = \left(\frac{K}{\tau s + 1}\right)A = \frac{AK}{\tau s + 1} = \frac{\frac{AK}{\tau}}{s + \frac{1}{\tau}}$$

$$L\left\{e^{-at}\right\} = \frac{1}{s + a}$$

$$c_i(t) = \frac{AK}{\tau} e^{-\frac{t}{\tau}} \qquad t \geq 0$$

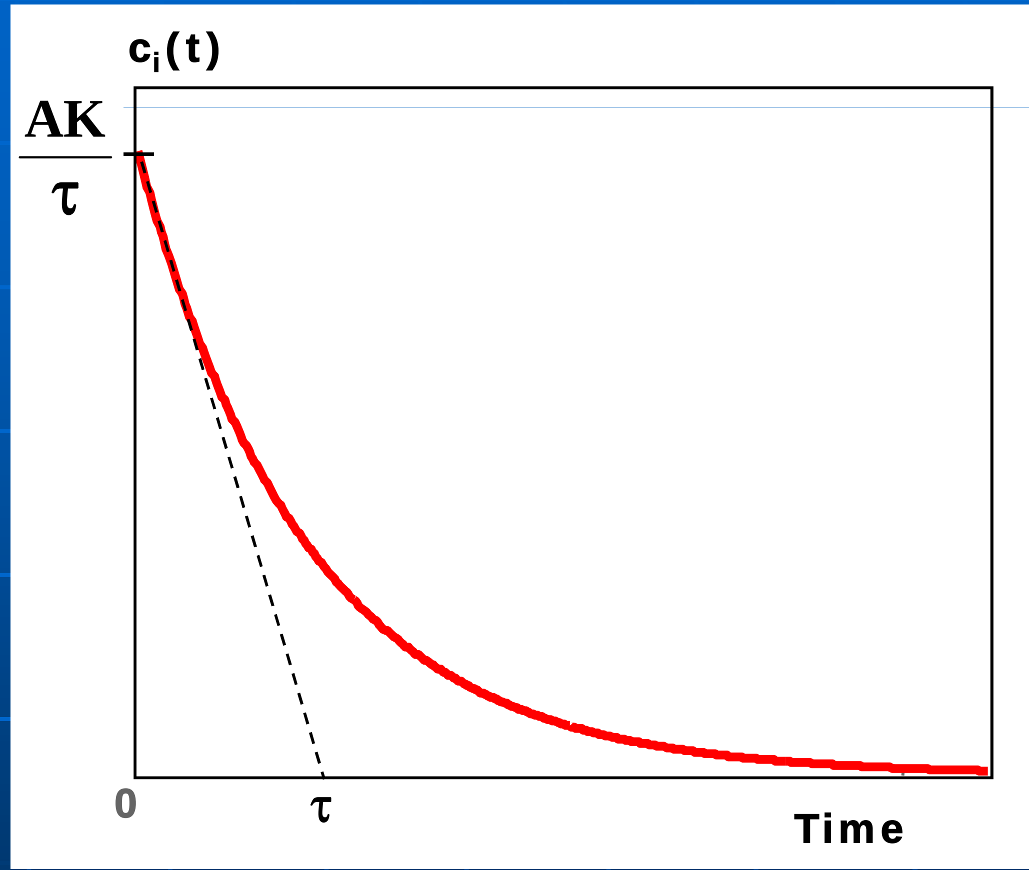
FIRST ORDER SYSTEMS

$$c_i(t) = \frac{AK}{\tau} e^{-\frac{t}{\tau}}$$

- Initial slope of the impulse response.

$$\frac{dc_i(t)}{dt} = -\frac{AK}{\tau^2} e^{-\frac{t}{\tau}}$$

$$\left.\frac{dc_i(t)}{dt}\right|_{t=0} = -\frac{AK}{\tau^2}$$



It is observed that the initial slope intersects the time axis at τ – How can this be useful ?

$$c_i(t) = \frac{AK}{\tau} e^{-\frac{t}{\tau}}$$

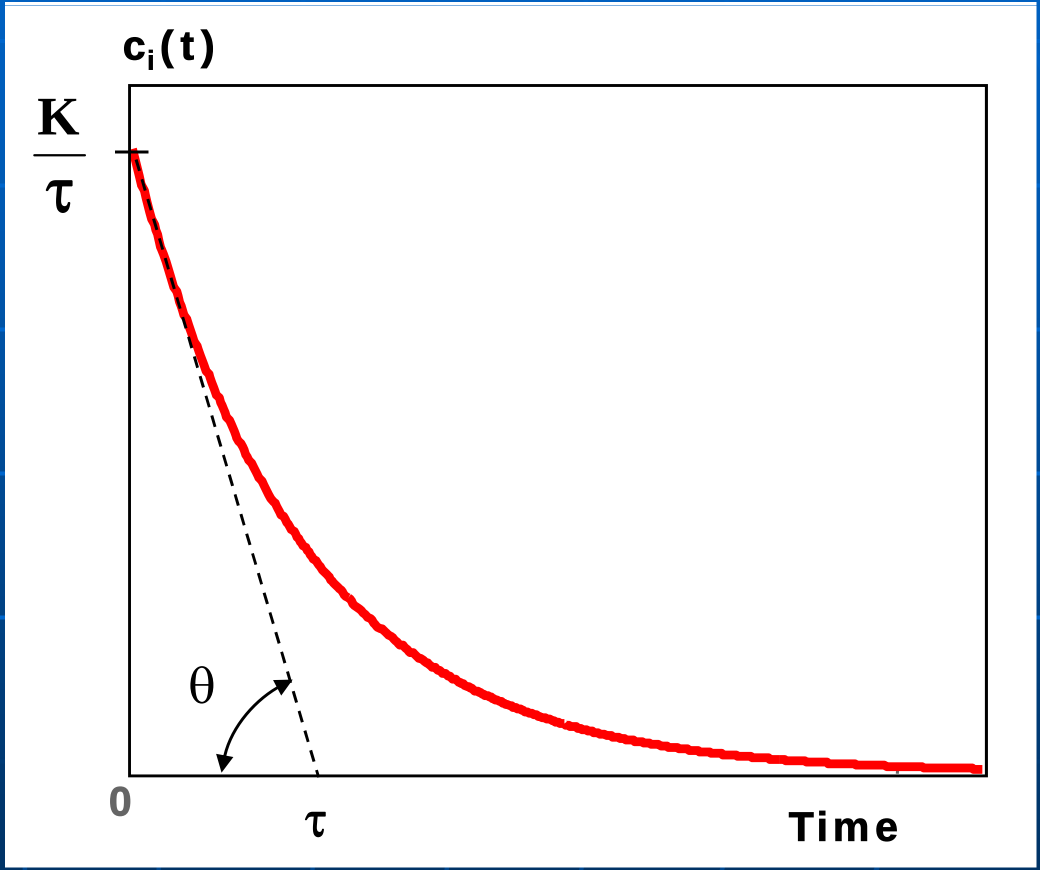
$$\left.\frac{dc_i(t)}{dt}\right|_{t=0} = -\frac{AK}{\tau^2}$$

FIRST ORDER SYSTEMS

- **Identification of a first order system from the unit impulse (A=1) response :**

- **Determine τ from the <u>initial slope</u> of the response.**

- **Determine K from the <u>initial value</u> of the response.**



$$R_s(s) = \frac{A}{s}$$

$$G(s) = \left(\frac{K}{\tau s + 1}\right)$$

FIRST ORDER SYSTEMS

- The **step response** (response to a step function) of the general first order system is :

$$C(s)=G(s)R(s)$$

$b = 1/\tau$

$$C_s(s) = \left(\frac{K}{\tau s + 1}\right)\frac{A}{s} = \frac{AK}{s(\tau s + 1)} = \frac{\frac{AK}{\tau}}{s\left(s + \frac{1}{\tau}\right)}$$

$$L\left\{\frac{1}{b}\left(1 - e^{-bt}\right)\right\} = \frac{1}{s(s+b)}$$

$$c_s(t) = \left(\frac{AK}{\tau}\right)\left\{\tau\left(1 - e^{-\frac{t}{\tau}}\right)\right\} = AK\left(1 - e^{-\frac{t}{\tau}}\right) \qquad t \geq 0$$

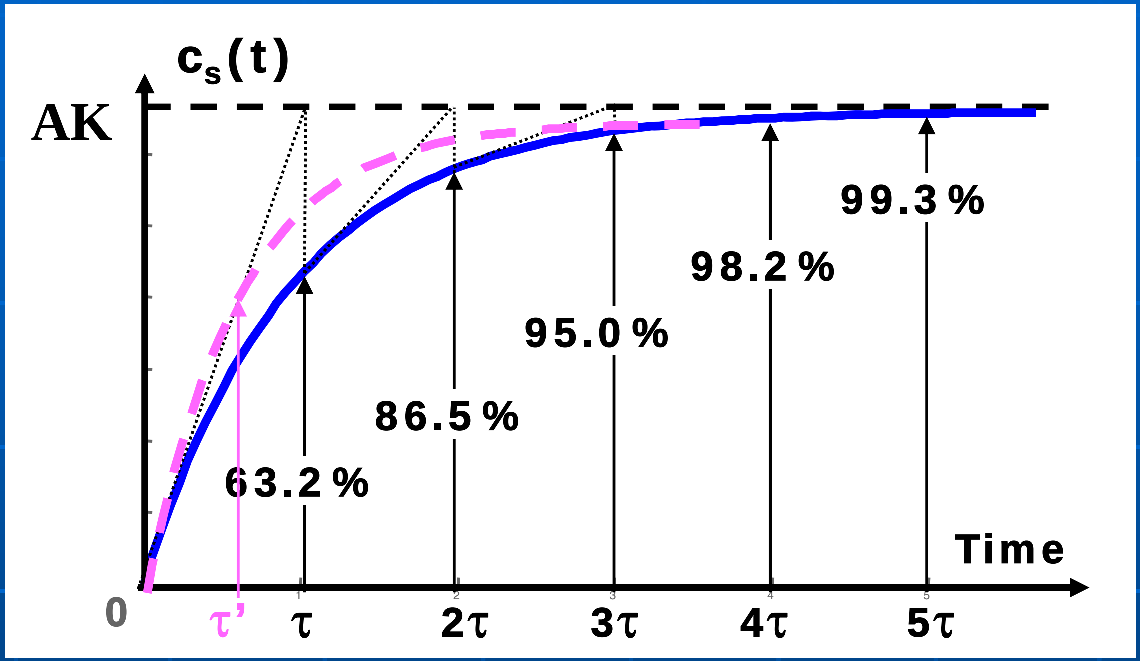
$$c_s(t) = AK\left(1 - e^{-\frac{t}{\tau}}\right) \qquad t \geq 0$$

FIRST ORDER SYSTEMS

- Initial slope of the step response.

$$\frac{dc_s(t)}{dt} = \frac{AK}{\tau} e^{-\frac{t}{\tau}}$$

$$\left.\frac{dc_s(t)}{dt}\right|_{t=0} = \frac{AK}{\tau}$$



- Observe that smaller time constant results in a faster system response (final value is approached in shorter time duration).
- Further, time response reaches within 2 and 1 % of the final value (AK) after 4 and 5 time constants, respectively.

$$c_s(t) = K\left(1 - e^{-\frac{t}{\tau}}\right) \qquad t \geq 0$$

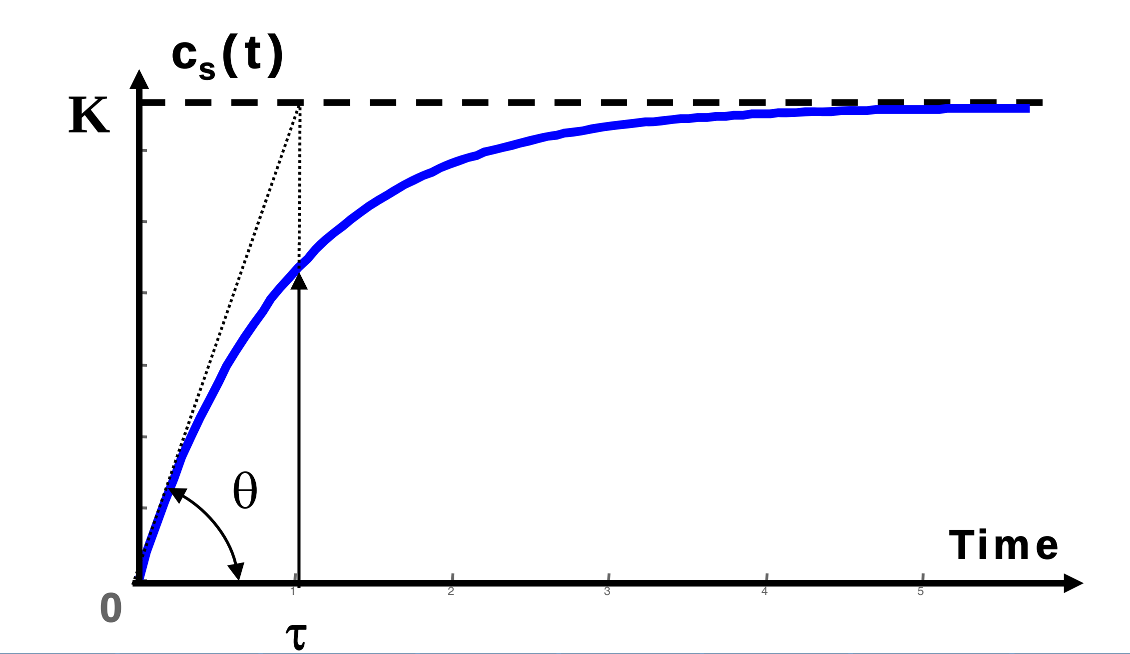
FIRST ORDER SYSTEMS

$$\left.\frac{dc_s(t)}{dt}\right|_{t=0} = \frac{K}{\tau}$$

- **Identification of a first order system from the unit step response :**

- **Determine K, which is the final value.**

- **Determine τ, which is the initial slope $\tan\theta = K/\tau$ of the response.**



$$R_r(s) = \frac{A}{s^2}$$

$$G(s) = \left(\frac{K}{\tau s + 1}\right)$$

FIRST ORDER SYSTEMS

- The **ramp response** (response to a ramp function) of the general first order system is :

C(s)=G(s)R(s)

$$C_r(s) = \left(\frac{K}{\tau s + 1}\right)\frac{A}{s^2} = \frac{AK}{s^2(\tau s + 1)} = \frac{\frac{AK}{\tau}}{s^2\left(s + \frac{1}{\tau}\right)}$$

$$C_r(s) = \frac{\left(\frac{AK}{\tau}\right)}{s^2\left(s + \frac{1}{\tau}\right)} = \frac{AK}{s^2} - \frac{AK\tau}{s} + \frac{AK\tau}{s + \frac{1}{\tau}}$$

Expand $C_r(s)$ into partial fractions .

FIRST ORDER SYSTEMS

- Expanding $C_r(s)$ into partial fractions and taking the inverse Laplace transform of each term the ramp response is obtained.

$$C_r(s) = \frac{\left(\frac{AK}{\tau}\right)}{s^2\left(s+\frac{1}{\tau}\right)} = \frac{AK}{s^2} - \frac{AK\tau}{s} + \frac{AK\tau}{s+\frac{1}{\tau}}$$

$a = 1/\tau$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$c_r(t) = AK(t-\tau) + AK\tau e^{-\frac{t}{\tau}} \qquad t \geq 0$$

$$c_r(t) = AK(t-\tau) + AK\tau e^{-\frac{t}{\tau}}$$

FIRST ORDER SYSTEMS

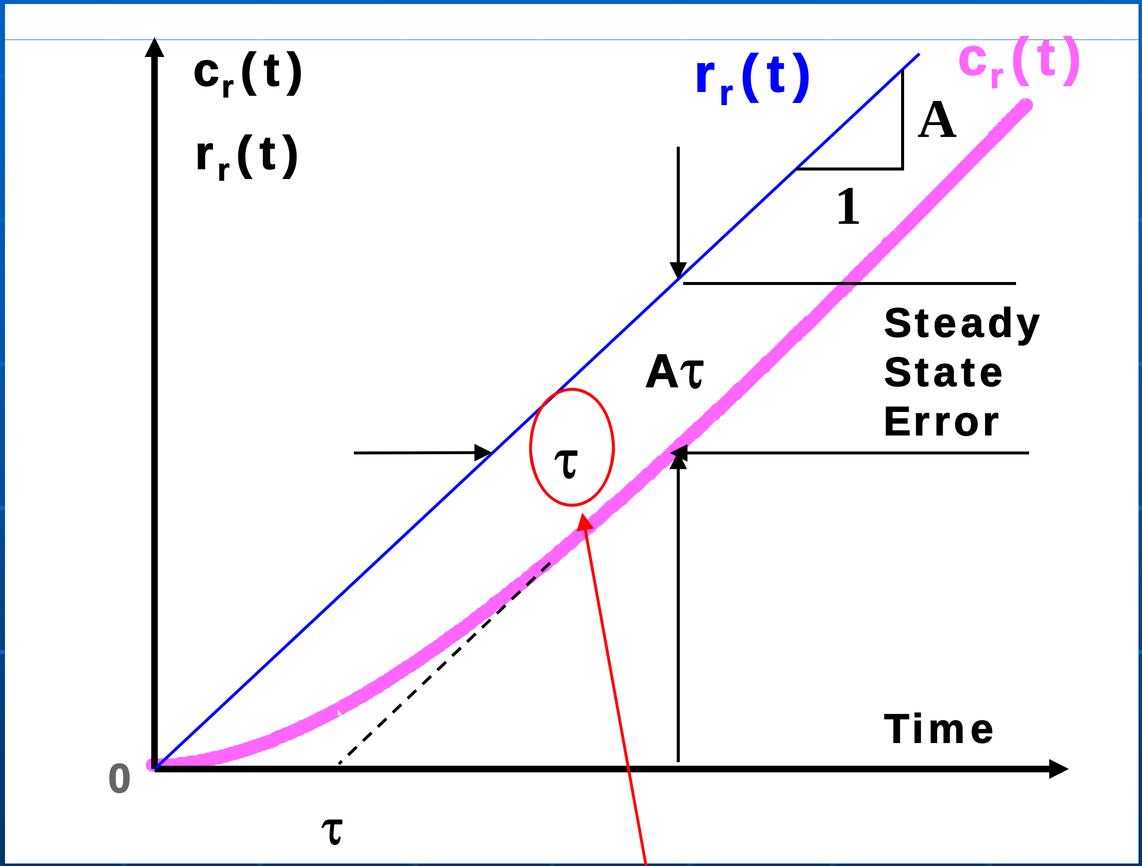
- For the special case of K=1:

$$e_r(t) = r_r(t) - c_r(t)$$

$$= At - At + A\tau - A\tau e^{-\frac{t}{\tau}}$$

$$e_r(t) = A\tau\left(1 - e^{-\frac{t}{\tau}}\right)$$

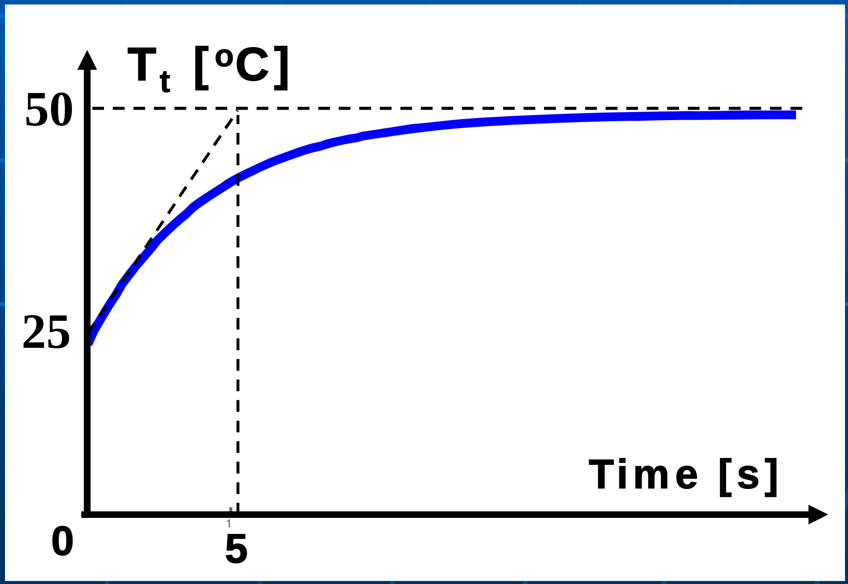
- As time goes to infinity, the error goes to Aτ.



Should this be Aτ ?

EXAMPLE 1a (See also Ogata p.210)

- A thin glass-walled thermometer, stabilized at the ambient temperature is suddenly immersed into a bath of water kept at a constant uniform temperature and a plot of the thermometer reading is shown in the figure.



EXAMPLE 1b

- Model the thermometer as a first order system. Using the experimental results, identify the thermometer dynamics, i.e. determine the gain and time constants of the transfer function.
- Then using the model, estimate the thermometer reading after 8 seconds if dipped now into a bath at a temperature of 80º C.

EXAMPLE 1c

- Model the thermometer as a thermal capacitance. Heat entering the thermometer from the bath is stored in it.

$$q = C_t \frac{dT_t}{dt}$$

- The thermal resistance or the thermometer frame is represented by :

$$T_b - T_t = R_t q$$

- Eliminate q from the two equations :

T_t : Thermometer reading
T_b : Bath temperature

$$T_b - T_t = R_t C_t \frac{dT_t}{dt}$$

EXAMPLE 1d

- One should note here that this system equation is in terms of the variable T_t which has nonzero initial value.

  Therefore, if the time response equation obtained by the application of the transfer function approach is to be used, new variables with zero initial condition must be defined.

$$\theta_t = T_t - T_0 \qquad \theta_b = T_b - T_0$$

EXAMPLE 1e

$$\theta_b = T_b - T_0 \qquad \theta_t = T_t - T_0$$

$$T_b - T_t = R_t C_t \frac{dT_t}{dt}$$

$$(\theta_b + T_0) - (\theta_t + T_0) = R_t C_t \frac{d(\theta_t + T_0)}{dt}$$

$$R_t C_t \frac{d\theta_t}{dt} + \theta_t = \theta_b$$

$$G(s) = \frac{\theta_t(s)}{\theta_b(s)} = \frac{1}{R_t C_t s + 1}$$

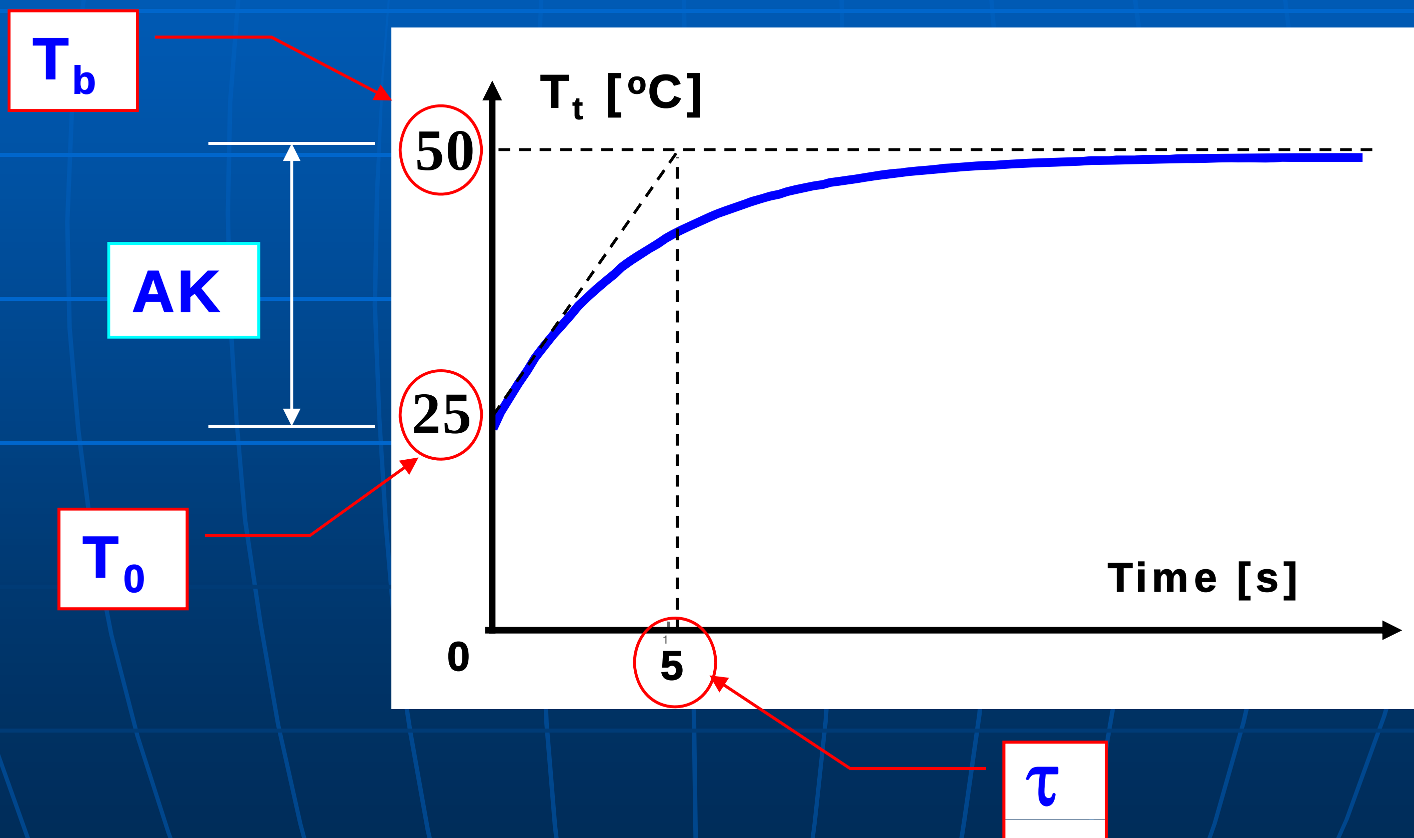
- **From the transfer function :**
  $\tau = R_t C_t$
  $K = 1$

EXAMPLE 1f

- From the plot, the value of τ is needed.

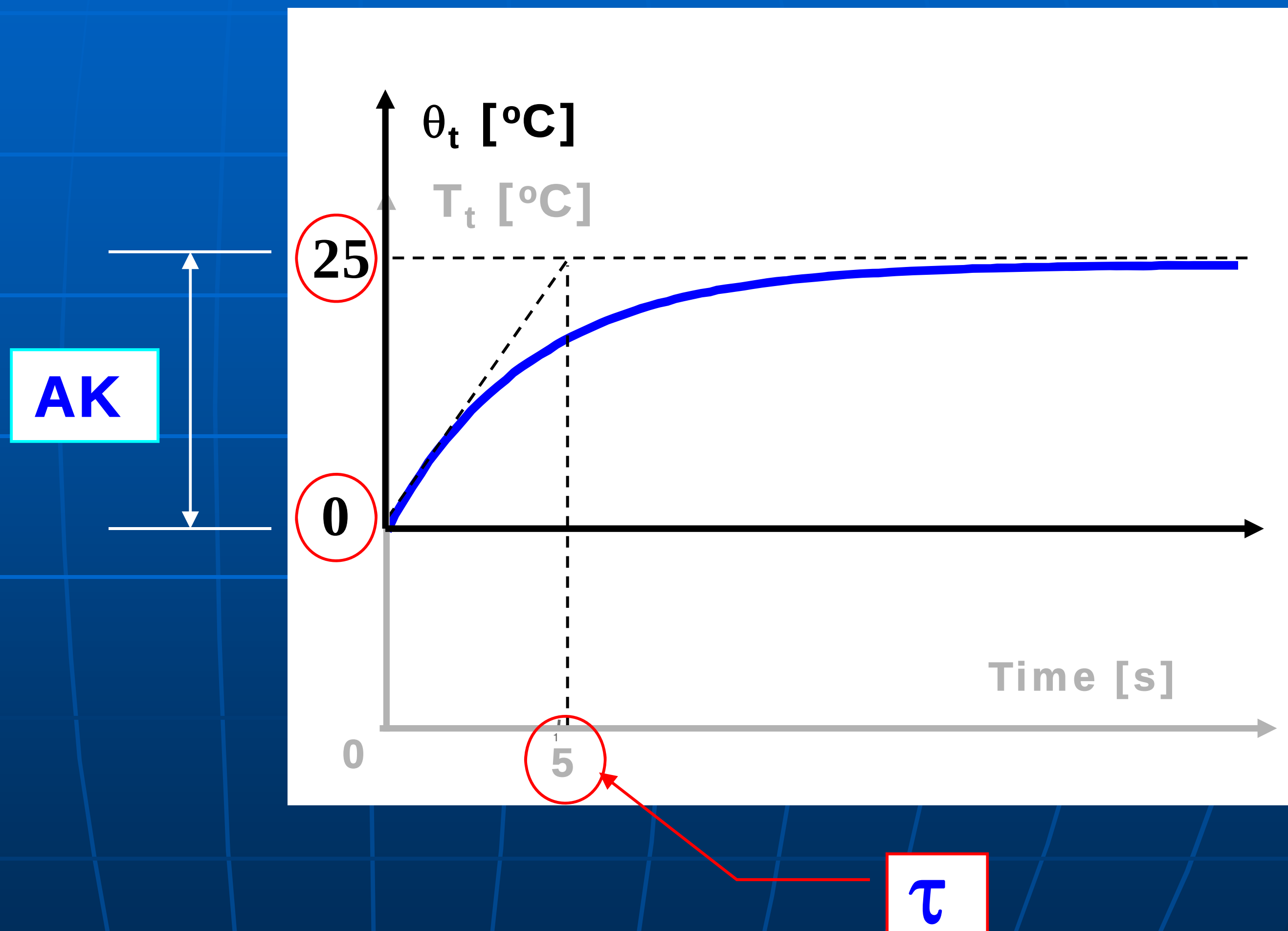
  $\tau = 5$

- The value of K is already known : K=1.



EXAMPLE 1g

- In reality, redefining variables with zero initial conditions is equivalent to a shift of coordinates of the plot.



EXAMPLE 1h

- When the bath temperature is 80ºC :

  $T_0 = 25^oC$

  $\theta_b = T_b - T_0 =$
  $= 80 - 25 = 55^oC$

  $AK = 55^oC$,

  $\tau = 5$ [s]

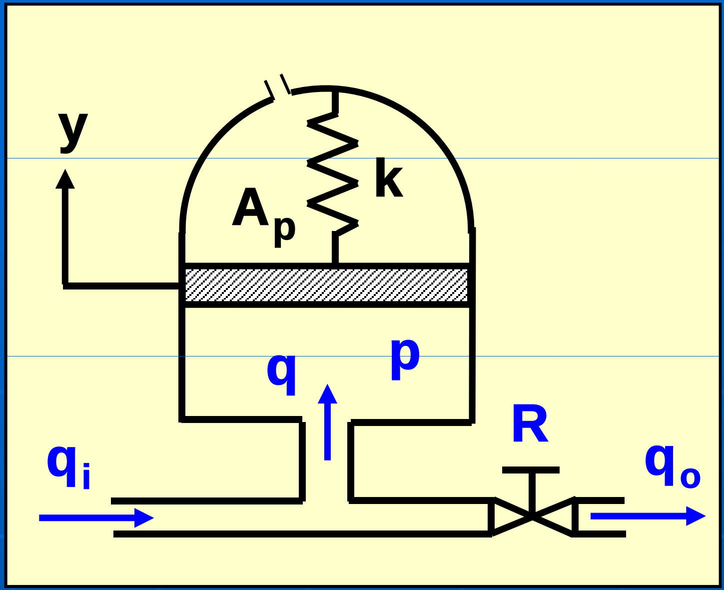
  $t = 8$ [s].

$$\theta_t = AK\left(1 - e^{-\frac{t}{\tau}}\right) = 55\left(1 - e^{-\frac{8}{5}}\right)$$

$$\theta_t = 43.9^o C$$

$$T_t = \theta_t + T_0 = 43.9 + 25 = 68.9^o C$$

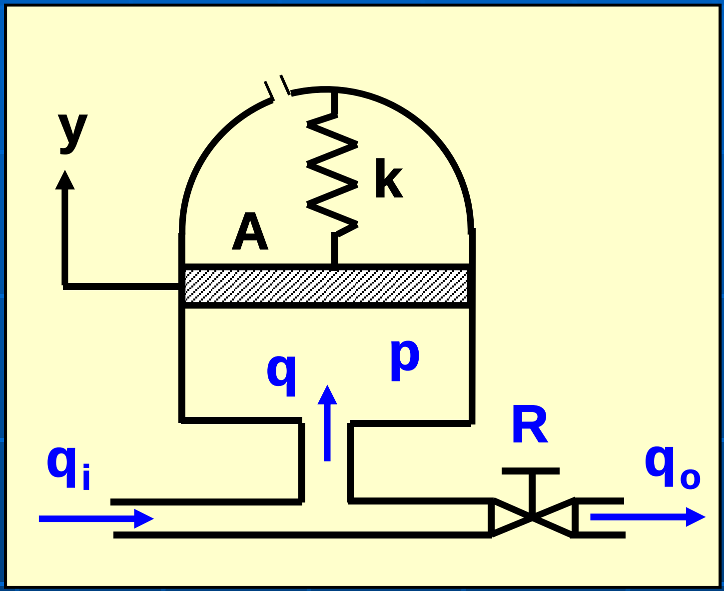
EXAMPLE 2a



- A hydraulic accumulator is illustrated in the figure. It is used to damp out pressure pulses by storing fluid during pressure peaks and releasing fluid during periods of low pressure.
- Neglect any friction as well as any pressure drop at the junction. The **massless** piston has an area A_p; the spring constant is **k**, and **y** is the displacement of the piston with respect to the **equilibrium position**.

EXAMPLE 2b

- Consider the hydraulic accumulator illustrated in the figure.
- Assume that the system is initially at steady state, i.e. the pressure inside the chamber and the flow rate in and out of the accumulator are constant at :

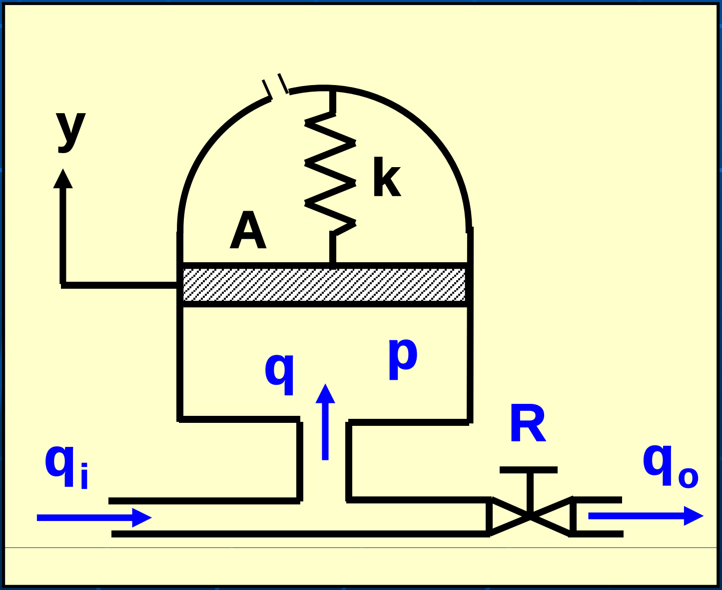


$\bar{p}$, $\bar{q}_i$, and $\bar{q}_o$

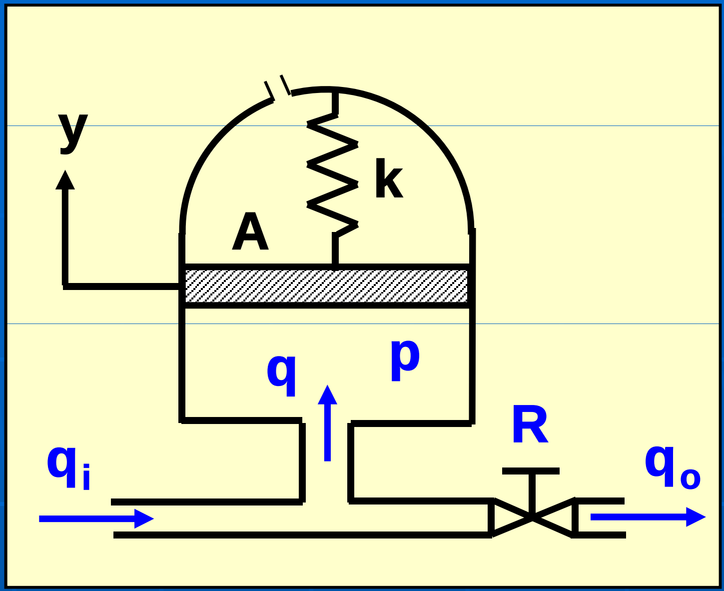
EXAMPLE 2c

- Determine the maximum pressure increase in the chamber, if the flow rate at the inlet suddenly increases from $2.5x10^{-4}$ to $1x10^{-3}$ m^3/s for 0.01 seconds and then goes back to its original value. Compare the maximum pressure ratio with and without the accumulator.

- The piston diameter is specified to be 5 cm, the spring constant is 500 N/m, and the valve resistance is 10^8 Ns/m^5.



EXAMPLE 2d



- For the <u>steady state</u> conditions :
- Elemental eqn. for the valve

$$R\bar{q}_o = \bar{p}$$

- Force balance on the mass

$$A_p\bar{p} = k\bar{y}$$

- Continuity Equations

$$\bar{q}_i - \bar{q} = \bar{q}_o$$

$$\bar{q} = A_p\frac{d\bar{y}}{dt} = 0$$

$$\bar{q}_i = \bar{q}_o$$

$R\bar{q}_o = \bar{p}$

$A_p\bar{p} = k\bar{y}$

$\bar{q}_i = \bar{q}_o$

$\bar{q} = 0$

EXAMPLE 2e

- **For the increased flow rate :**

$$R\left(\bar{q}_o + q_o\right) = \left(\bar{p} + p\right)$$

$$Rq_o = p$$

$$A_p\left(\bar{p} + p\right) = k\left(\bar{y} + y\right)$$

$$A_p p = ky$$

$$\left(\bar{q}_i + q_i\right) - \left(\bar{q} + q\right) = \left(\bar{q}_o + q_o\right)$$

$$q_i - q = q_o$$

$$\left(\bar{q} + q\right) = A_p \frac{d\left(\bar{y} + y\right)}{dt}$$

$$q = A_p \frac{dy}{dt}$$

EXAMPLE 2f

- **Take the Laplace transforms :**

$Rq_o = p$	$\Rightarrow$	$RQ_o(s) = P(s)$
$A_p p = ky$		$A_p P(s) = kY(s)$
$q_i - q = q_o$		$Q_i(s) - Q(s) = Q_o(s)$
$q = A_p \dfrac{dy}{dt}$		$Q(s) = A_p sY(s)$

$$RQ_o(s) = P(s)$$

$$A_p P(s) = kY(s)$$

$$Q_i(s) - Q(s) = Q_o(s)$$

$$Q(s) = A_p sY(s)$$

EXAMPLE 2g

- Eliminate $Q_o(s)$, $Q(s)$, and $Y(s)$ from these equations.

$$G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$$

⟺

$$G(s) = \frac{P(s)}{Q_i(s)} = \frac{R}{\frac{RA_p^2}{k}s + 1}$$

- Compare with the standard form :

$$K = R$$

$$\tau = \frac{RA_p^2}{k}$$

EXAMPLE 2h

- The flow rate at the inlet suddenly increases from $2.5x10^{-4}$ to $1x10^{-3}$ m^3/s for 0.01 seconds and then goes back to its original value.
- Since the input is of very short duration, it can be approximated by an **impulse** with :

magnitude = $1x10^{-3} - 2.5x10^{-4} = 7.5x10^{-4}$ m^3/s

duration = 0.01 s, and

strength = $(7.5x10^{-4}\ m^3/s)(0.01\ s)$

= $7.5x10^{-6}$ m^3

$$A = 7.5x10^{-6}\ m^3$$

$$A = 7.5\text{x}10^{-6}\,\text{m}^3$$

EXAMPLE 2i

- The piston diameter is specified to be 5 cm, the spring constant is 500 N/m, and the valve resistance is 10^8 Ns/m^5.

$$A_p = \frac{\pi}{4}(0.05)^2 = 0.00196\,\text{m}^2$$

$$K = R = 10^8\,\text{Ns/m}^5$$

$$\tau = \frac{RA_p^2}{k} = \frac{\left(10^8\,\frac{\text{Ns}}{\text{m}^5}\right)\left(0.00196\text{m}^2\right)^2}{500\,\frac{\text{N}}{\text{m}}} \cong 0.77[\text{s}]$$

$$A = 7.5x10^{-6}\,m^3$$

$$K = 10^8\,Ns/m^5$$

$$\tau = 0.77\,[s]$$

EXAMPLE 2j

- The impulse response of a first order system is given by :

$$c_i(t) = \frac{AK}{\tau} e^{-\frac{t}{\tau}}$$

$$p(t) = \frac{\left(7.5x10^{-6}\,m^3\right)\left(10^8\,\frac{Ns}{m^5}\right)}{0.77\,s} e^{-\frac{t}{0.77}} = 974e^{-1.3t}\left[\frac{N}{m^2}\right]$$

$$p_{max}(t=0) = 974\left[\frac{N}{m^2}\right] = 0.974\left[\frac{kN}{m^2}\right]$$

$$p_{max}(t=0) = 0.974\left[\frac{kN}{m^2}\right]$$

EXAMPLE 2k

- The steady state pressure in the chamber was:

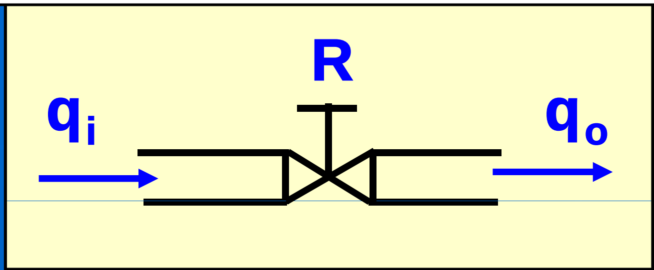
$$\bar{p} = R\bar{q}_o = R\bar{q}_i = \left(10^8 \frac{Ns}{m^5}\right)\left(2.5x10^{-4}\frac{m^3}{s}\right) = 25\left[\frac{kN}{m^2}\right]$$

- The ratio of the maximum pressure to steady state pressure is :

$$\frac{\bar{p} + p_{max}}{\bar{p}} = 1 + \frac{p_{max}}{\bar{p}} = 1 + \frac{0.974}{25} \cong 1.04$$

$$\frac{\bar{p} + p_{max}}{\bar{p}} = 1.04$$

EXAMPLE 2l



- Now let us consider the case without the accumulator. At steady state :

$$R\bar{q}_o = \bar{p} \qquad \bar{q}_i = \bar{q}_o$$

- After the jump in flow rate :

$$R\left(\bar{q}_o + q_o\right) = \left(\bar{p} + p\right)$$

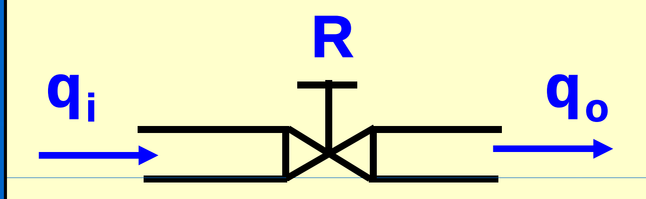
$$\left(\bar{q}_i + q_i\right) = \left(\bar{q}_o + q_o\right)$$

$$Rq_o = p$$

$$q_i = q_o$$

$$\frac{\bar{p}+p_{max}}{\bar{p}} = 1.04$$

EXAMPLE 2m



- Steady state pressure in the case without the accumulator :

$$p = Rq_{i_{max}} = \left(10^8 \frac{Ns}{m^5}\right)\left(0.75x10^{-3} \frac{m^3}{s}\right) = 75\left[\frac{kN}{m^2}\right]$$

- The ratio of the maximum pressure to steady state pressure is :

$$\frac{\bar{p}+p_{max}}{\bar{p}} = 1 + \frac{p_{max}}{\bar{p}} = 1 + \frac{75}{25} = 4$$

- The benefit of using an accumulator in reducing pressure peaks is obvious.

$$G(s) = \frac{X(s)}{Y(s)}$$

Response to Initial Conditions
(Ogata p.263-264)

- The general first order transfer function represents the differential equation

$$a_1 \dot{x} + a_0 x = b_0 y$$

which may have the initial condition : $x(0) = x_0$

Taking the Laplace transform of the terms of the differential equation, removing the input y and keeping the initial condition :

$$a_1 \left[sX(s) - x_0 \right] + a_0 X(s) = 0$$

Reorder the equation.

$$\left(a_1 s + a_0 \right) X(s) = a_1 x_0$$

$$\left(a_1 s + a_0\right) X(s) = a_1 x_0$$

Response to Initial Conditions

Thus the response of the first order system to initial is the same as the **unit step response** of the system with the TF in the parenthesis,

$$X(s) = \left[\frac{a_1 x_0 s}{\left(a_1 s + a_0\right)}\right]\left(\frac{1}{s}\right)$$

or the **unit impulse response** of the system with the TF in the parenthesis :

$$X(s) = \left[\frac{a_1 x_0}{\left(a_1 s + a_0\right)}\right](1)$$