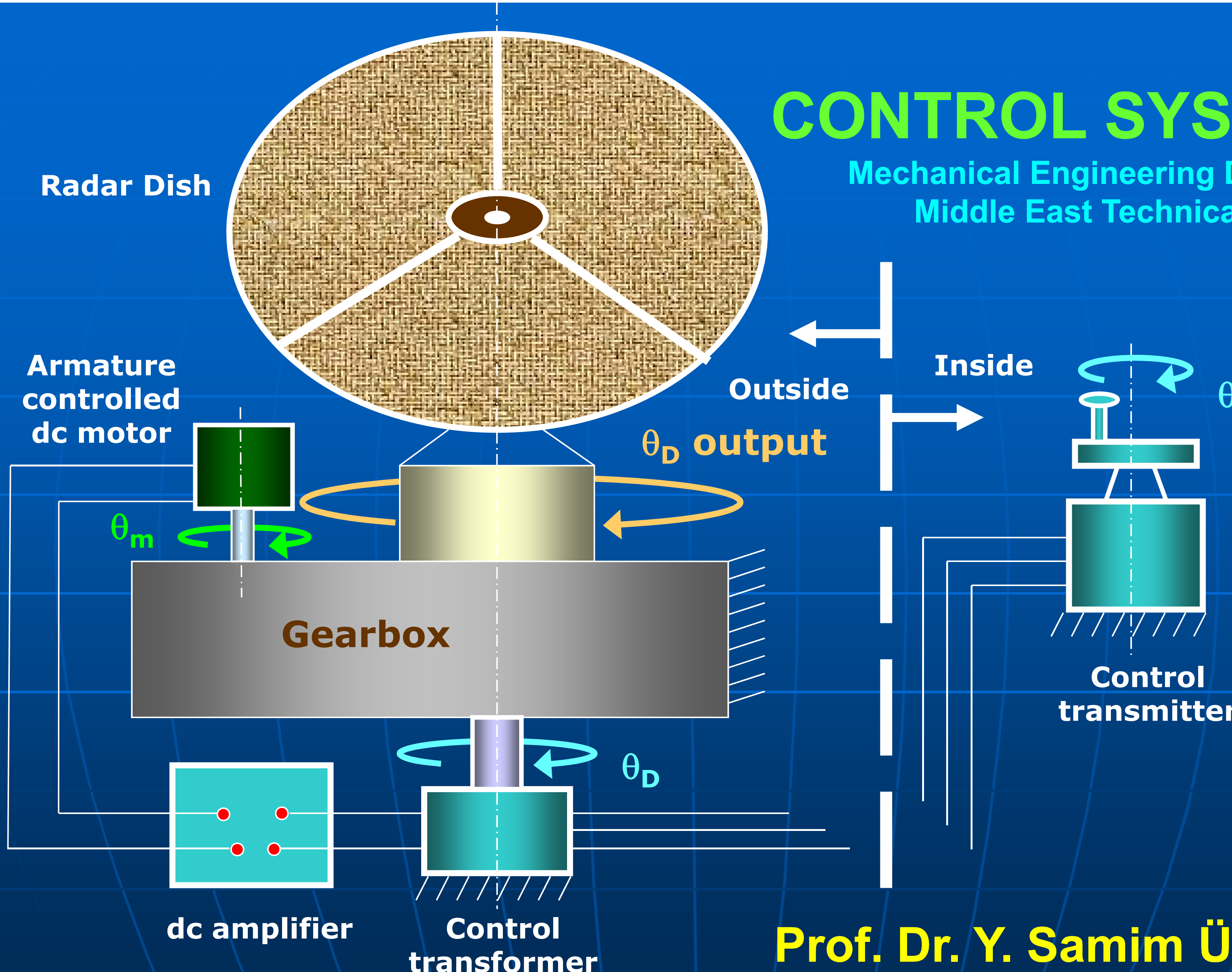
ME 304

CONTROL SYSTEMS

Mechanical Engineering Department,
Middle East Technical University



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CH IV

COURSE OUTLINE

I. INTRODUCTION & BASIC CONCEPTS
II. MODELING DYNAMIC SYSTEMS
III. CONTROL SYSTEM COMPONENTS
IV. STABILITY
V. TRANSIENT RESPONSE
VI. STEADY STATE RESPONSE
VII. DISTURBANCE REJECTION
VIII. BASIC CONTROL ACTIONS & CONTROLLERS
IX. FREQUENCY RESPONSE ANALYSIS
X. SENSITIVITY ANALYSIS
XI. ROOT LOCUS ANALYSIS

STABILITY - OBJECTIVES

In this chapter :

- Formal definitions of stability will be examined and the relation between the roots of the characteristic equation and the stability of the system will be examined.
- Routh's stability criterion will be applied to various control systems.
- Selection of controller parameters for stable response will be illustrated.
- The concepts of relative stability and stability margin will be introduced.

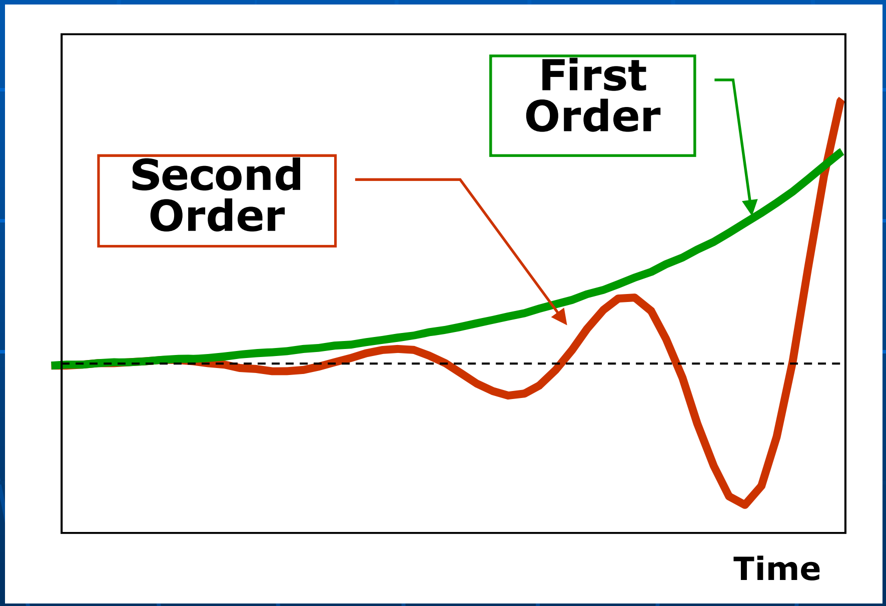
DEFINITIONS OF STABILITY

Nise Ch. 6.1-6.4, Dorf&Bishop Ch. 6, Ogata 5.7

- **Bounded-Input, Bounded-Output Stability (Zero state response) : A linear time invariant system is said to be stable if it produces a bounded response to a bounded input.**
- **Zero Input and Asymptotic Stability (Zero input response) : A system is stable, if the zero input response due to finite initial conditions returns to zero asymptotically as time goes to infinity.**

DEFINITIONS OF STABILITY

- Thus for an unstable system, the response will increase without bounds or will never return to the equilibrium state.



TRANSFER FUNCTION

- General form of the **transfer function** :

$$G(s)=\frac{X(s)}{Y(s)}=\frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}=\frac{N(s)}{D(s)}$$

- n : order of the system ($n \geq m$),
- D(s) : characteristic polynomial.
- Characteristic equation : D(s)=0

$$D(s)=a_n s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 = 0$$

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

TRANSFER FUNCTION

- The roots of the numerator polynomial, i.e.

$$N(s) = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0 = 0$$

are called the zeroes of the system.

- The roots of the denominator polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

are called the poles of the system.

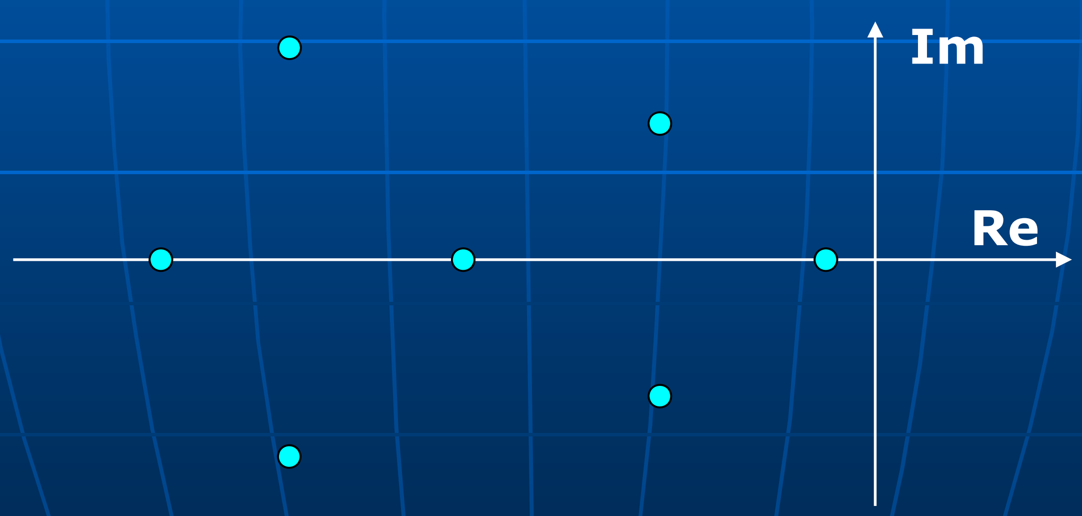
STABILITY and Poles

- Stability of a LTI system is a property of the system and is independent of the inputs.
- It can be shown that the positions of the roots of the characteristic equation (poles of the transfer function) in the complex plane determine the stability of the system.

$$y(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} \left[A_k e^{-\xi_k \omega_k t} \cos\left(\omega_k \sqrt{1-\xi_k^2}\, t - \phi_k \right) \right]$$

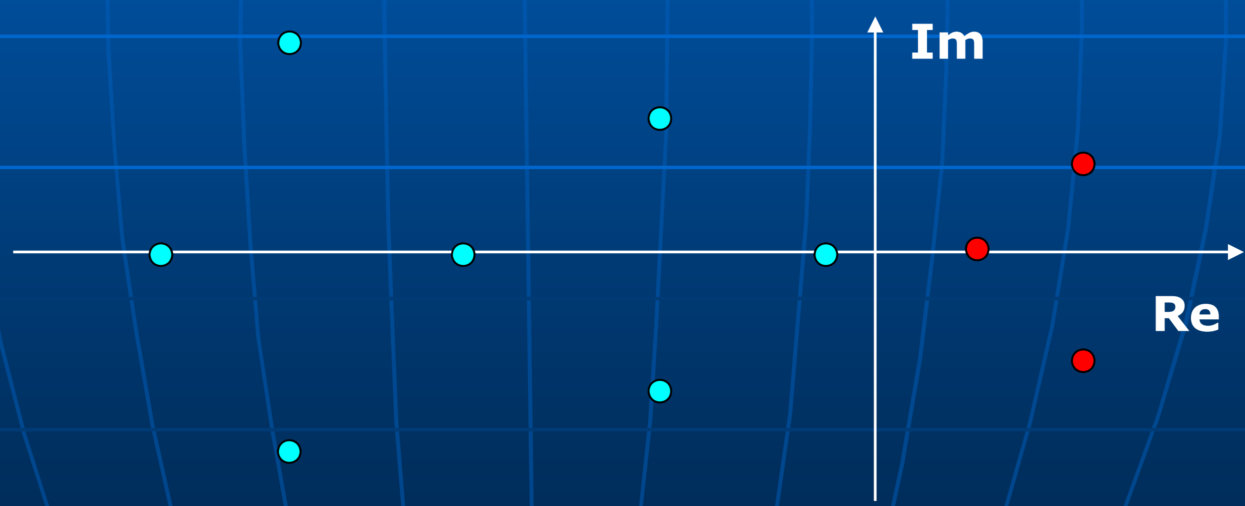
STABILITY and Poles

- If all the roots of the characteristic equation are on the left hand side of the complex plane, i.e. all the roots have negative real parts, then the system is stable.



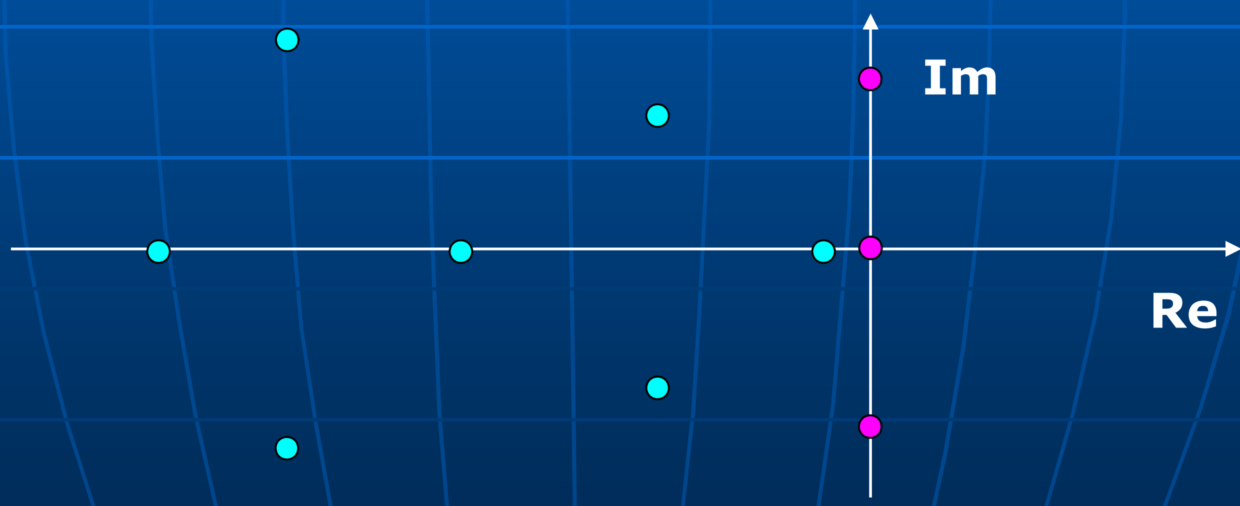
STABILITY and Poles

- **If there is at least one root on the right hand side of the complex plane, then the system is unstable and the response will increase without bounds with time.**



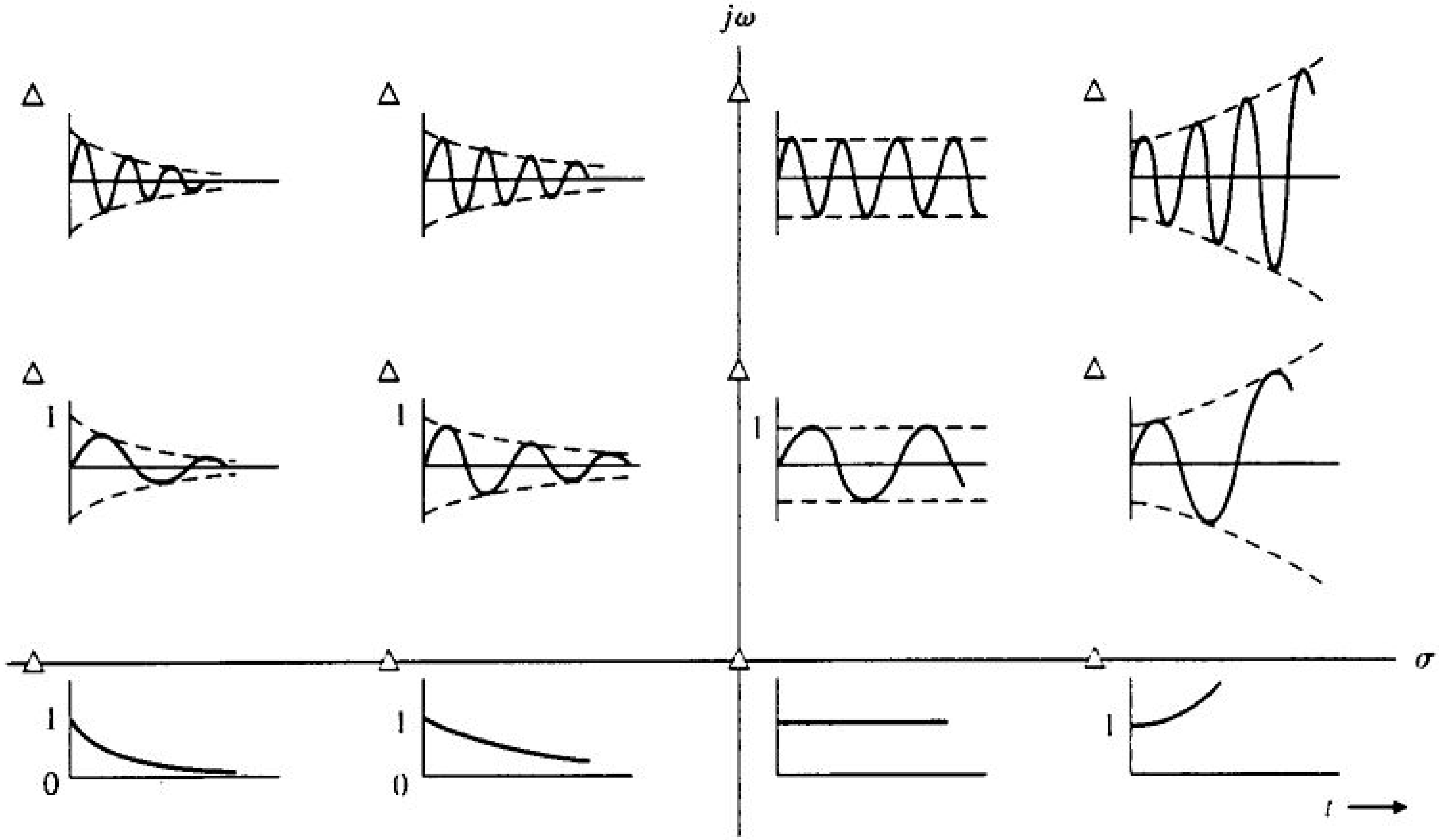
STABILITY and Poles

- If there is at least one root with zero real part, i.e. on the imaginary axis, then the response will contain undamped sinusoidal oscillations or a nondecaying response.
- If there are no unstable roots, the response neither decreases to zero, nor increases without bounds. The system is called **marginally stable**.



STABILITY

$r = \sigma + j\omega$



STABILITY and Poles

- The direct approach to the determination of the stability of a system would therefore be the calculation of the roots of the characteristic equation.
- The calculation of the roots of the characteristic equation is not possible or practical, however, if parameter values are not yet available, and conditions on these parameters for a stable system are to be obtained.

ROUTH'S STABILITY CRITERION

Nise 6.2, Dorf & Bishop 6.2, Ogata pp. 275-281

- **Routh's stability criterion allows the determination of**
  - **whether there are any roots of the characteristic equation with positive real parts**

and, if there are,

- **the number of these roots**

without actually finding the roots.

ROUTH'S STABILITY CRITERION

- **The first step in checking the stability of a system using Routh's stability criterion is the application of an initial test called the Hurwitz test.**

- **Hurwitz Test :**

  The necessary but not sufficient condition for a characteristic equation

$$D(s)=a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0=0$$

  to have all its roots with negative real parts is that all of the coefficients a_i must exist and have the same sign.

ROUTH'S STABILITY CRITERION

- **If the characteristic equation fails to meet the above condition, then the system is <u>not</u> stable.**

$$D(s)=3s^4+s^3+4s^2+5$$

$$D(s)=s^5+4s^3+3s^2+24s-12$$

- **If, however, the condition is satisfied, then <u>no conclusion on the stability of the system can be reached</u> !**

$$D(s)=3s^5+2s^4+s^3+6s^2+7s+2$$

ROUTH'S STABILITY CRITERION

- If Hurwitz condition is satisfied, then Routh's stability criterion must be used to determine the stability of the system.
- To be able to apply Routh's criterion, Routh's array must be constructed.
- For a real polynomial

$$D(s)=a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

the Routh's array is a special arrangement of the coefficients in a certain pattern.

ROUTH'S STABILITY CRITERION

- **Routh's array :** $D(s)=a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	. . .
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	. . .
s^{n-2}	b_1	b_2	b_3	. . .
s^{n-3}	c_1	c_2	c_3	. . .
.	.			
.	.			
.	.			
s^2	e_1			
s^1	f_1			
s^0	g_1			

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1}$$

ROUTH'S STABILITY CRITERION

- **Example :** $D(s)=s^3+20s^2+9s+100$

Passes Hurwitz test !

s^3	1	9
s^2	20	100
s^1	4	
s^0	100	

$$b_1 = \frac{(20)(9)-(1)(100)}{20} = 4$$

$$c_1 = \frac{(4)(100)-(20)(0)}{4} = 100$$

Knight's move (chess)

ROUTH'S STABILITY CRITERION

- **Routh's Stability Criterion :**

  The necessary and sufficient condition for a characteristic equation to have all its roots with negative real parts is that the elements of the first column of the Routh's array to have the same sign.

  If the elements of the first column have different signs, then the number of sign changes is equal to the number of roots with positive real parts.

ROUTH'S STABILITY CRITERION

- **Example :** $D(s)=s^3+s^2+2s+24$

Passes Hurwitz test !

$$s_1 = -3.0000$$
$$s_2 = 1.0000 + 2.6458i$$
$$s_3 = 1.0000 - 2.6458i$$

s^3	1	2
s^2	1	24
s^1	-22	0
s^0	24	

$$b_1 = \frac{(1)(2)-(1)(24)}{1} = -22$$

$$c_1 = \frac{(-22)(24)-(1)(0)}{-22} = 24$$

Knight's move

Sign changes in the 1st column : Unstable system.

2 sign changes : two roots with positive real parts.

ROUTH'S STABILITY CRITERION

- **Special Cases :**

  There are some cases in which problems appear in completing the Routh's array.

  They are encountered in the case of systems that are not stable, and means are devised to allow the completion of the Routh's array.

ROUTH'S STABILITY CRITERION

Nise 6.3

- **Special Case 1 :**

  When a first column term in a row becomes zero with all other terms being nonzero, the calculation of the rest of the terms becomes impossible due to division by zero.

  In such a case the system is <u>unstable</u> and the procedure is continued just to determine the number of roots with positive real parts.

ROUTH'S STABILITY CRITERION

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0
s^2	???		
s^1			
s^0			

$$b_1 = \frac{(2)(2)-(1)(4)}{2} = 0$$

$$b_2 = \frac{(2)(11)-(1)(10)}{2} = 6$$

$$c_1 = \frac{(0)(4)-(2)(6)}{0}$$

ROUTH'S STABILITY CRITERION

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0
s^2	???		
s^1			
s^0			

In such a case, replace zero term by a very small and positive number ε.

ROUTH'S STABILITY CRITERION

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

s^5	1	2	11
s^4	2	4	10
s^3	$0 \rightarrow \varepsilon$	6	0
s^2	$-\frac{12}{\varepsilon}$	10	0
s^1	6	0	
s^0	10		

$$c_1 = \frac{(4)(\varepsilon)-(2)(6)}{\varepsilon} = 4 - \frac{12}{\varepsilon}$$

$$\varepsilon \rightarrow 0 \quad \Rightarrow \quad c_1 \cong -\frac{12}{\varepsilon}$$

$$d_1 = \frac{\left(-\frac{12}{\varepsilon}\right)(6)-(10)(\varepsilon)}{-\frac{12}{\varepsilon}} = 6 + \frac{10}{12}\varepsilon^2$$

$$\varepsilon \rightarrow 0 \quad \Rightarrow \quad d_1 \cong 6$$

ROUTH'S STABILITY CRITERION

- **Special Case 1 :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

s^5	1	2	11
s^4	2	4	10
s^3	$0 \rightarrow \varepsilon$	6	0
s^2	$-\frac{12}{\varepsilon}$	10	
s^1	6		
s^0	10		

2 sign changes : 2 roots with positive real part.

$s_1 = 0.8950 + 1.4561i$

$s_2 = 0.8950 - 1.4561i$

$s_3 = -1.2407 + 1.0375i$

$s_4 = -1.2407 - 1.0375i$

$s_5 = -1.3087$

ROUTH'S STABILITY CRITERION

- **Special Case 2 :**

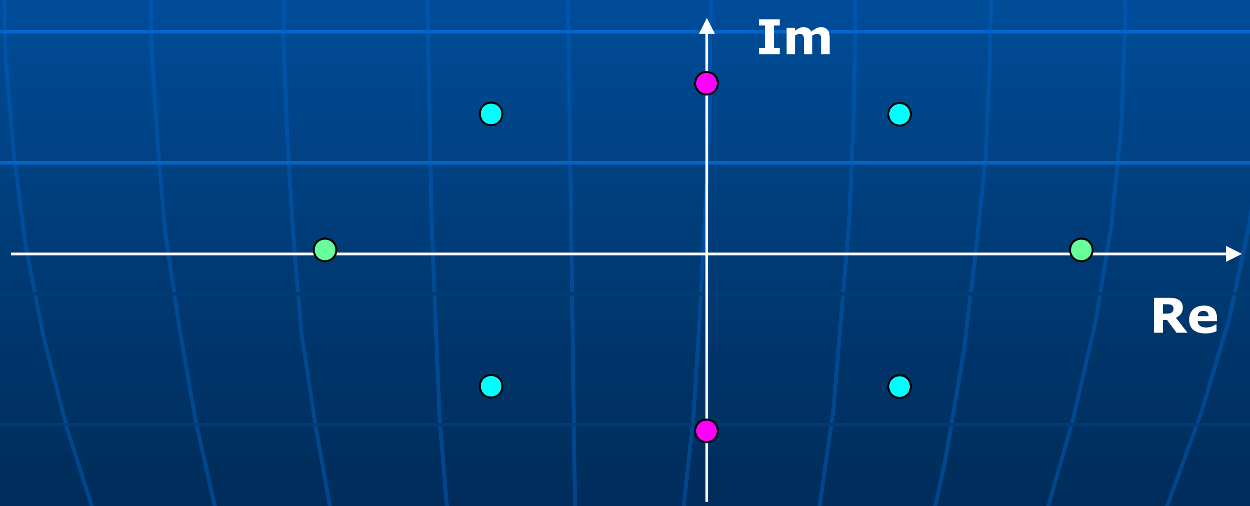
  If all the terms on a derived row are zero, this means that the characteristic equation has roots which are symmetric with respect to the origin,

  1. **Two real roots with equal magnitudes but opposite signs, and/or**
  2. **Two conjugate imaginary roots, and/or**
  3. **Two complex roots with equal real and imaginary parts of opposite signs.**

ROUTH'S STABILITY CRITERION

- <u>**Special Case 2 :**</u>

  In such a case the system is <u>not stable</u> and the procedure is continued to determine if it is marginally stable or unstable, and in the second case to determine the number of roots with positive real parts.



ROUTH'S STABILITY CRITERION

- **Special Case 2 :**

  To proceed, an auxiliary polynomial Q(s) is formed by using the terms of the row just before the row of zeros. The auxiliary polynomial Q(s) is always even (i.e. all powers of s are even !).

  The roots of Q(s)=0 will give the symmetric roots of the characteristic polynomial.

  To complete the Routh's array, simply replace the row of zeroes with the coefficients of dQ(s)/ds=0.

ROUTH'S STABILITY CRITERION

- **Example :** $D(s)=s^3+2s^2+s+2$ **Passes Hurwitz test !**

s^3	1	1
s^2	2	2
s^1	0	0
	4	0
s^0	2	

$$Q(s)=(2)s^2+(2)s^0$$

Replace row of zeroes with dQ/ds

$$\frac{dQ(s)}{ds}=(4)s+(0)s^0$$

No sign changes in the 1st column :
No roots with positive real parts.

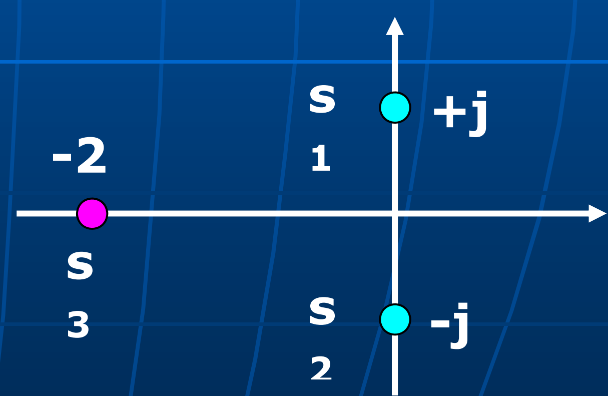
ROUTH'S STABILITY CRITERION

- **Example :** $D(s)=s^3+2s^2+s+2$ **Passes Hurwitz test !**

s^3	1	1
s^2	2	2
s^1	0	0
	4	0
s^0	2	

$$Q(s)=2s^2+2=0 \qquad s_{1,2}=\pm j$$

$$\frac{s^3+2s^2+s+2}{2s^2+2}=\frac{s}{2}+1 \qquad s_3=-2$$



STABILITY - OBJECTIVES

In this chapter :

- Formal definitions of stability will be examined and the relation between the roots of the characteristic equation and the stability of the system will be examined.
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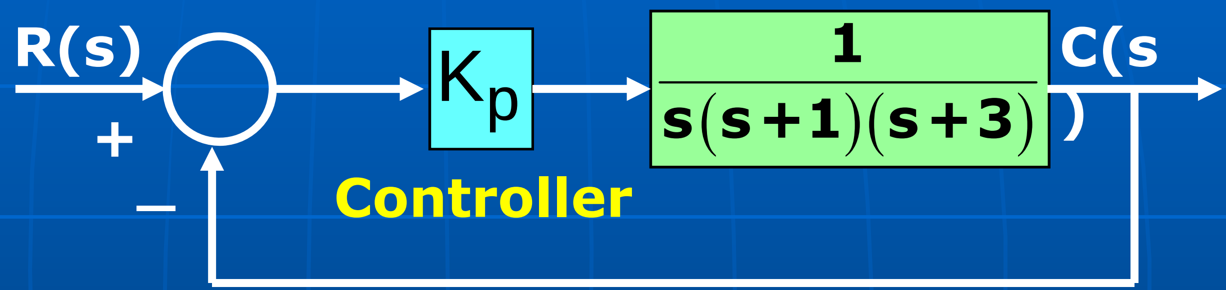
We are here !

STABILITY BOUNDARIES

- One of the steps in the design and optimization of control systems is the selection of controller parameters.
- The limiting values of these parameters leading to instability must be determined first, so that best values in the stable range can be chosen.

EXAMPLE 1a
Nise 6.4

- Determine the range of values for the controller parameter K_p for which the system will be stable.



- First determine the characteristic polynomial.

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{K_p}{s(s+1)(s+3)}}{1+\dfrac{K_p}{s(s+1)(s+3)}}=\frac{K_p}{s(s+1)(s+3)+K_p}=\frac{K_p}{s^3+4s^2+3s+K_p}$$

$$D(s)=s^3+4s^2+3s+K_p$$

EXAMPLE 1b

- $D(s)=s^3+4s^2+3s+K_p$

s^3	1	3
s^2	4	K_p
s^1	$3-\frac{K_p}{4}$	0
s^0	K_p	

For stability :

$$3-\frac{K_p}{4}>0 \quad \text{and} \quad K_p>0$$

$$0<K_p<12$$

EXAMPLE 1c

- $D(s)=s^3+4s^2+3s+K_p$

s^3	1	3
s^2	4	K_p
s^1	$3-\frac{K_p}{4}$	0
s^0	K_p	

For $K_p = 0$:

$s(s+1)(s+3)=0$

$s_1 = 0,\ s_2 = -1,\ s_3 = -3$

Marginally stable!

Non-oscillatory response.

For $K_p = 12$:

$(s+4)(s^2+3)=0$

$s_1 = -4,\ s_2 = +\sqrt{3}j,\ s_3 = -\sqrt{3}j$

Marginally stable!

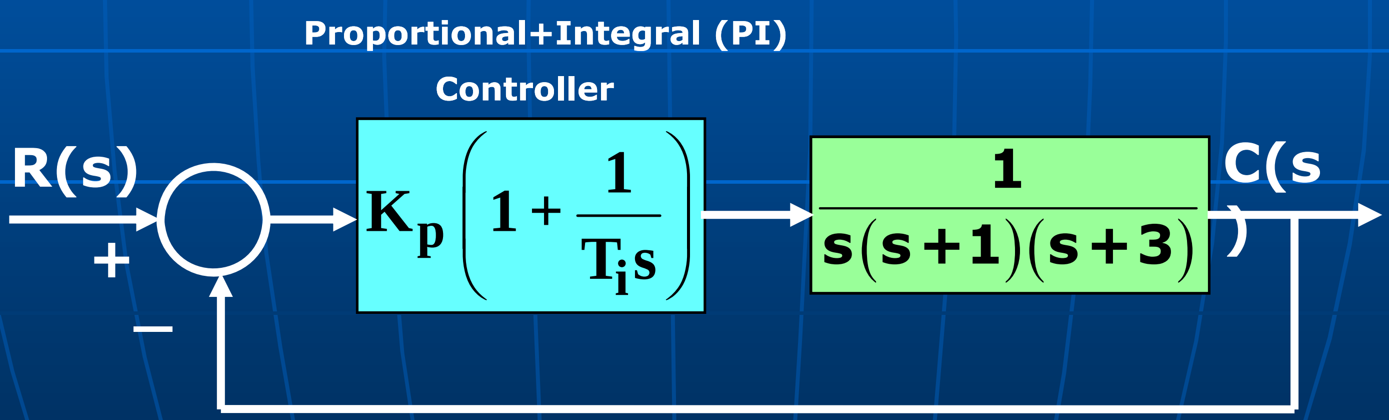
Oscillatory response,

Undamped oscillations.

What is the frequency of undamped oscillations ?

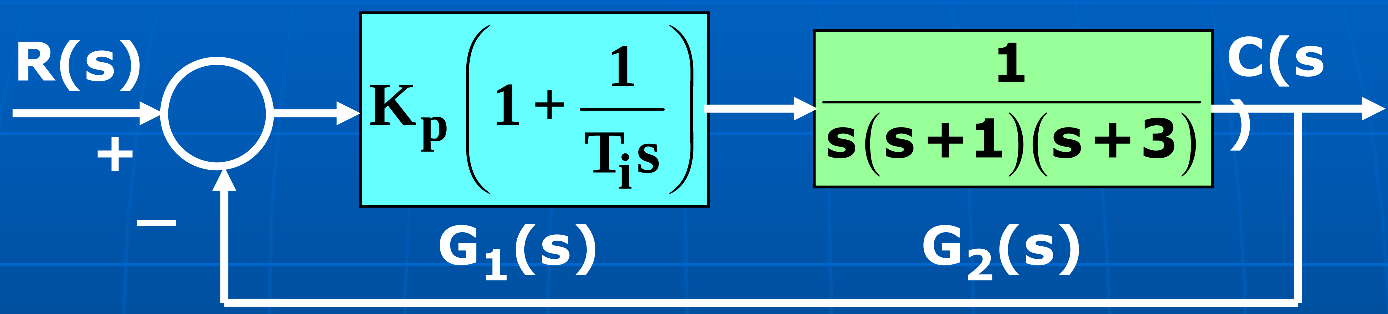
STABILITY BOUNDARIES – Example 2a

- For the control system represented by the block diagram shown, determine and sketch the regions of stability with respect to the controller parameters K_p and T_i.



STABILITY BOUNDARIES – Example 2b

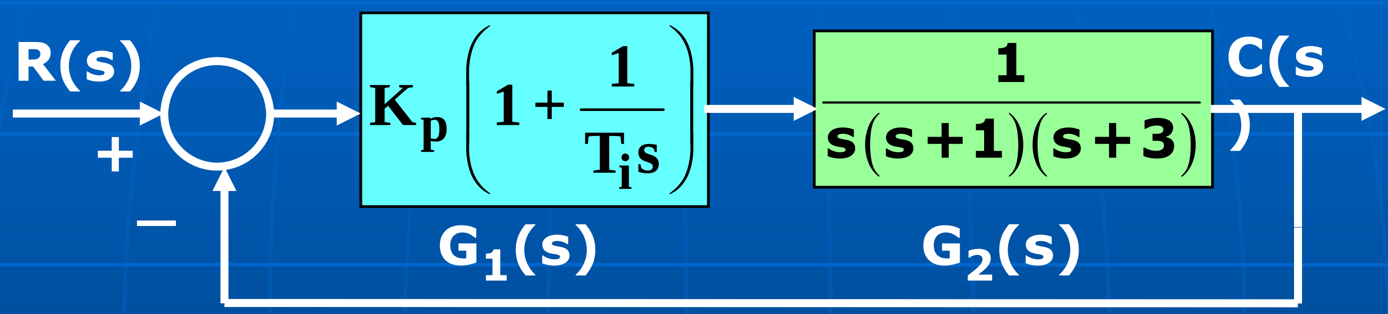
- **Determine the characteristic polynomial first.**



$$G(s)=\frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)}=\frac{K_p\left(1+\frac{1}{T_i s}\right)\left(\frac{1}{s(s+1)(s+3)}\right)}{1+K_p\left(1+\frac{1}{T_i s}\right)\left(\frac{1}{s(s+1)(s+3)}\right)}$$

STABILITY BOUNDARIES – Example 2c

- **Determine the characteristic polynomial first.**



$$G(s) = \frac{K_p\left(T_i s + 1\right)}{T_i s^2\left(s+1\right)\left(s+3\right) + K_p\left(T_i s + 1\right)}$$

$$D(s) = T_i s^4 + 4T_i s^3 + 3T_i s^2 + K_p T_i s + K_p$$

STABILITY BOUNDARIES – Example 2d

- $D(s)=T_i s^4+4T_i s^3+3T_i s^2+K_pT_i s+K_p$

s^4	T_i	$3T_i$	K_p
s^3	$4T_i$	K_pT_i	0
s^2	$\left(3-\frac{K_p}{4}\right)T_i$	K_p	0
s^1	$\left(T_i-\frac{16}{12-K_p}\right)K_p$	0	0
s^0	K_p		

$T_i > 0$

$K_p > 0$

$3-\frac{K_p}{4} > 0 \quad \Rightarrow \quad K_p < 12$

$0 < K_p < 12$

$T_i - \frac{16}{12-K_p} > 0$

$T_i > \frac{16}{12-K_p}$

STABILITY BOUNDARIES – Example 2e

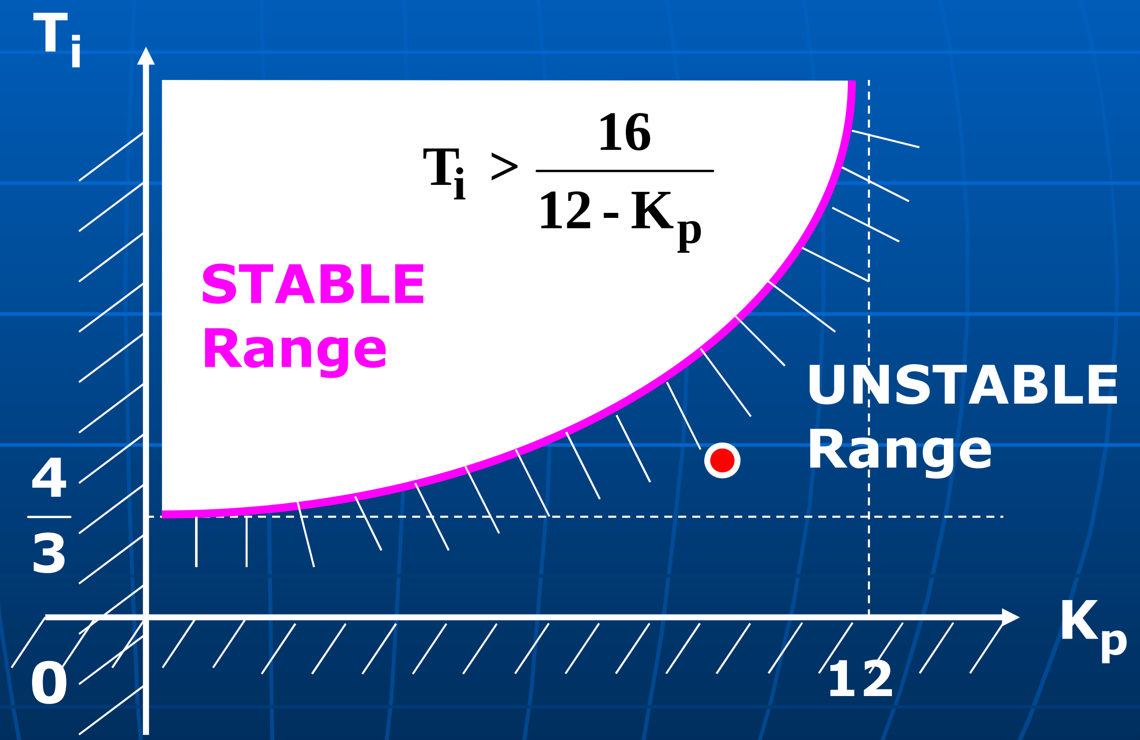
- $D(s)=T_i s^4+4T_i s^3+3T_i s^2+K_pT_i s+K_p$

Regions of stability in the parameter plane.

$$0 < K_p < 12$$

$$T_i > \frac{16}{12-K_p}$$

Check if a selection $K_p=10$, $T_i=2$ is acceptable !



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RELATIVE STABILITY & STABILITY MARGIN

Dorf & Bishop 6.3

- **Absolute stability** is related to the determination of whether a given system is stable or not.
- **Relative stability** is related to the distance, from the origin of the complex plane, of the real part of the root of the characteristic equation nearest to the imaginary axis of the complex plane.
- The absolute magnitude of the real part of the root closest to the origin of the complex plane is called the **stability margin**.

RELATIVE STABILITY & STABILITY MARGIN

- In many practical applications, determination of absolute stability is not sufficient.
- Information related to relative stability is also required.
- The Routh's stability criterion can also be used to check if a system satisfies the requirement of a specified stability margin.

STABILITY MARGIN – Example 3a

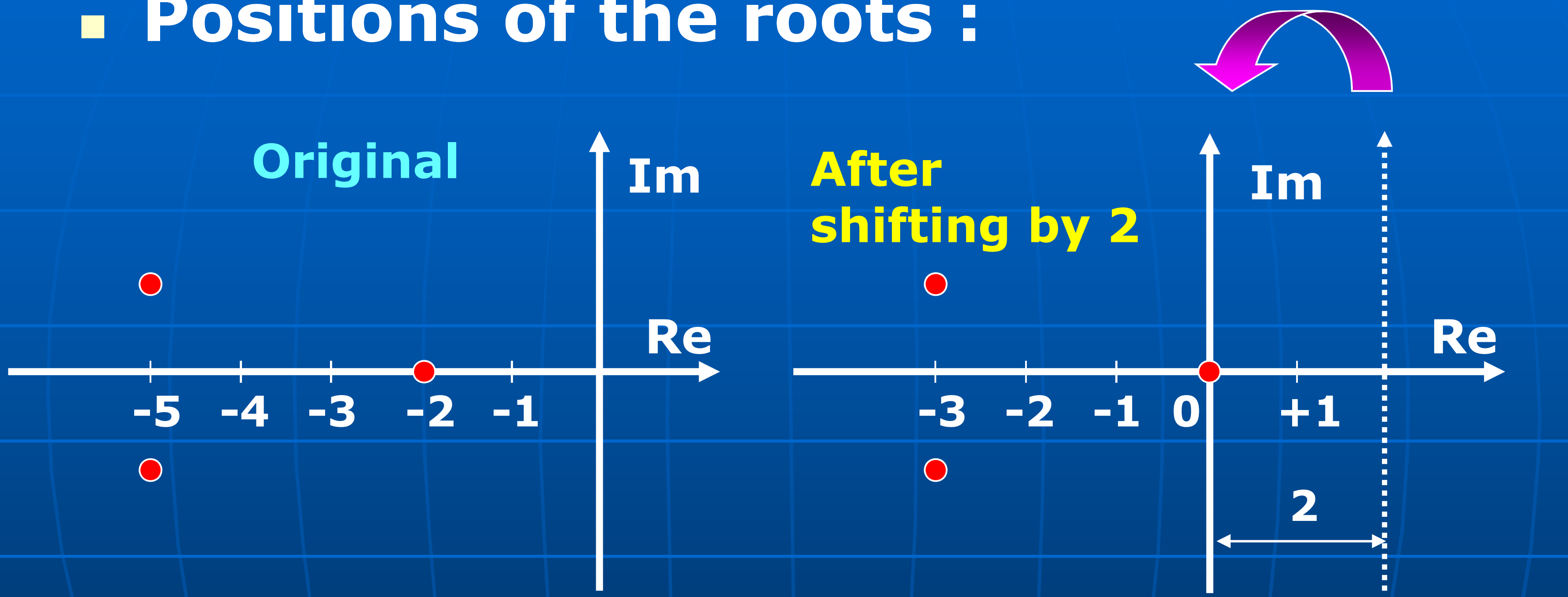
- **Determine if the system with the characteristic polynomial**

  $$D(s)=s^3+12s^2+46s+52$$

  can provide a stability margin of 2.1.
- **The approach in this problem is to shift the origin of the complex plane towards the left by 2.1.**
- **If the system is still stable, then the system has a stability margin greater than 2.1. If the system becomes unstable, then it has a stability margin of less than 2.1.**

STABILITY MARGIN – Example 3b

- **Positions of the roots :**



$$s_1 = -5.0000 + 1.0000i$$
$$s_2 = -5.0000 - 1.0000i$$
$$s_3 = -2.0000$$

$$s_1 = -3.0000 + 1.0000i$$
$$s_2 = -3.0000 - 1.0000i$$
$$s_3 = 0$$

STABILITY MARGIN – Example 3c

- **To shift the imaginary axis by 2.1 to the left, define a new complex variable**

  $$z = s + 2.1$$

  so that point -2.1 becomes the origin.

- **Hence introduce $s = z - 2.1$ in the characteristic equation.**

  $$D(z) = (z-2.1)^3 + 12(z-2.1)^2 + 46(z-2.1) + 52$$

  $$D(z) = z^3 + 5.7z^2 + 8.83z - 0.941$$

STABILITY MARGIN – Example 3d

$$D(z)=z^3+5.7z^2+8.83z-0.941$$

It is immediately obvious that the system has become unstable !

Thus the stability margin of the system is less than 2.1.

Now, we can check the number of roots causing this instability using the Routh's stability criterion.

STABILITY MARGIN – Example 3e

$$D(z)=z^3+5.7z^2+8.83z-0.941$$

z^3	1	8.83		Polynomial
z^2	5.7	-0.941		fails Hurwitz
z^1	8.995	0		test !
z^0	-0.941			

Since there is only one sign change, there is one root with a positive real part in the range from 0 to 2.1.