M E T U
Department of Mathematics

	Ideals, Varieties and Algorithms **Midterm 1**	

Code : *Math 473* Acad. Year : *2019-2020* Semester : *Fall* Instructor : *Tolga Karayayla* Date : *07.11.2019* Time : *17.40* Duration : *120 minutes*	Last Name : First Name : Student ID : Department : Signature : **6** Questions on **4** Pages SHOW DETAILED WORK!

1	2	3	4	5	6		

NOTE: k is a field in all questions below.

1. (15 pts.) Use *lex* order with $x > y$ and perform the Division Algorithm to divide $f = x^2y + 2y^3 + 3x^4y^5$ by (f_1, f_2, f_3) in $\mathbb{Q}[x, y]$ where $f_1 = x^3y^2 + x + 2$, $f_2 = xy^3 + 2y$ and $f_3 = y^2 + 3$.

$a_1 = 3xy^3$

$a_2 = -3x - 6$

$a_3 = 2y$

r

$f_1 = x^3y^2 + x + 2$

$f_2 = xy^3 + 2y$

$f_3 = y^2 + 3$

$$3x^4y^5 + x^2y + 2y^3$$
$$-\;3x^4y^5 + 3x^2y^3 + 6xy^3$$
$$-3x^2y^3 + x^2y - 6xy^3 + 2y^3$$
$$-\;-3x^2y^3 - 6xy$$
$$x^2y - 6xy^3 + 6xy + 2y^3 \longrightarrow x^2y$$
$$-6xy^3 + 6xy + 2y^3$$
$$-\;-6xy^3 - 12xy$$
$$6xy + 2y^3 + 12y \longrightarrow x^2y + 6xy$$
$$2y^3 + 12y$$
$$-\;2y^3 + 6y$$
$$6y \longrightarrow x^2y + 6xy + 6y$$
$$0$$

$f = a_1 \cdot f_1 + a_2 \cdot f_2 + a_3 f_3 + r$

where $a_1 = 3xy^3$, $a_2 = -3x - 6$, $a_3 = 2y$, $r = x^2y + 6xy + 6y$

None of terms of r are divisible by any of $LT(f_1)$, $LT(f_2)$ and $LT(f_3)$

2. (15 pts.) Fact: $\{g_1, g_2, ..., g_r\} \subset I$ is a Groebner basis of an ideal $I \subset k[x_1, x_2, ..., x_n]$ if and only if for every $f \in I$ there is at least one g_i such that $LT(g_i)$ divides $LT(f)$.
Use this fact to show that $\{g_1, g_2, g_3\}$ is not a Groebner basis of $I = \langle g_1, g_2, g_3\rangle \subset \mathbb{R}[x, y, z]$ in grlex order with $x > y > z$ where $g_1 = x^2y + y^2z^2 + z$, $g_2 = x^4 + 2xy^4z + 1$ and $g_3 = x^2y^2 + yz + 2$.

In grlex order, $LT(g_1) = y^2z^2$, $LT(g_2) = 2xy^4z$, $LT(g_3) = x^2y^2$

$f = x^2 \cdot g_1 - z^2 \cdot g_3 \Rightarrow f \in I = \langle g_1, g_2, g_3\rangle$

$= x^2(x^2y + y^2z^2 + z) - z^2(x^2y^2 + yz + 2) = x^4y + \cancel{x^2y^2z^2} + x^2z - \cancel{x^2y^2z^2} - yz^3 - 2z^2$

$= x^4y + x^2z - yz^3 - 2$, so $LT(f) = x^4y$

$y^2z^2 \nmid x^4y$, $xy^4z \nmid x^4y$, $x^2y^2 \nmid LT(f) = x^4y$

$LT(g_1) \nmid LT(f)$, $LT(g_2) \nmid LT(f)$, $LT(g_3) \nmid LT(f)$

$f \in I$, but $LT(f)$ is not divisible by any of $LT(g_1)$, $LT(g_2)$, $LT(g_3)$, then by the given "fact" above, $\{g_1, g_2, g_3\}$ is not a Groebner basis of $I = \langle g_1, g_2, g_3\rangle$ in grlex order.

3. (10+10 pts.) a) For any monomial order $>$ show that $LT(f \cdot g) = LT(f) \cdot LT(g)$ for any nonzero polynomials f and g in $k[x_1, x_2, ..., x_n]$.

Let $f = a_1 x^{\alpha(1)} + a_2 x^{\alpha(2)} + \dots + a_r x^{\alpha(r)} = \sum_{i=1}^{r} a_i x^{\alpha(i)}$ where $LT(f) = a_1 x^{\alpha(1)}$, $a_1 \neq 0$

$g = b_1 x^{\beta(1)} + \dots + b_s x^{\beta(s)} = \sum_{j=1}^{s} b_j x^{\beta(j)}$ where $LT(g) = b_1 \cdot x^{\beta(1)}$, $b_1 \neq 0$.

(Notation: $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$)

Then $f \cdot g = \left(\sum_{i=1}^{r} a_i x^{\alpha(i)}\right) \cdot \left(\sum_{j=1}^{s} b_j x^{\beta(j)}\right) = \sum_{i=1}^{r} \sum_{j=1}^{s} a_i b_j x^{\alpha(i)} \cdot x^{\beta(j)}$

$x^{\alpha(1)} \geq x^{\alpha(i)}$ and $x^{\beta(1)} \geq x^{\beta(j)}$ for all i and j

Then $x^{\alpha(1)} \cdot x^{\beta(1)} \geq x^{\alpha(i)} \cdot x^{\beta(1)}$ for all i
$x^{\beta(1)} \cdot x^{\alpha(i)} \geq x^{\beta(j)} \cdot x^{\alpha(i)}$ for all i and j
$\Rightarrow x^{\alpha(1)} \cdot x^{\beta(1)} \geq x^{\alpha(i)} \cdot x^{\beta(1)} \geq x^{\alpha(i)} \cdot x^{\beta(j)}$
$x^{\alpha(1)} \cdot x^{\beta(1)} \geq x^{\alpha(i)} \cdot x^{\beta(j)}$ for all i and j (equality holds only if $(i,j) = (1,1)$)

$a_1 b_1 \neq 0$, so $a_1 b_1 x^{\alpha(1)} \cdot x^{\beta_1} = LT(f) \cdot LT(g)$ is the term of $f \cdot g$ with the largest monomial, hence it is $LT(f \cdot g)$

b) Let $I = \langle g\rangle \subset k[x_1, ..., x_n]$ where $g \neq 0$ be a principal ideal. Show that if G is a finite subset of I containing cg for some $c \in k - \{0\}$, then G is a Groebner basis of I.

Let $cg \in G$, $G \subseteq I$, G is finite. $(c \in k - \{0\})$

Let $f \in I = \langle g\rangle$ and $f \neq 0$, then $f = h \cdot g$ for some $h \in k[x_1, \dots, x_n]$

Then $LT(c \cdot g) = LT(c) \cdot LT(g) = c \cdot LT(g)$ ($LT(c) = c$ since $c \in k - \{0\}$, constant

$LT(f) = LT(h \cdot g) = LT(h) \cdot LT(g)$

$c \cdot LT(g) \mid LT(h) \cdot LT(g)$ since $LT(h) LT(g) = c \cdot LT(g) \cdot \frac{1}{c} \cdot LT(h)$ ($\frac{1}{c} \in k$ if $c \neq 0$ in k)

$cg \in G$, $LT(c \cdot g) \mid LT(f)$ for any $f \in I$.

Then by the "Fact" in question 2, G is a Groebner basis of $I = \langle g\rangle$

4. (3 × 9 pts.) a) Let $W = \{(t^5, t^3, t^2) \in k^3 | t \in k\}$. Show that $W = V(x^3 - y^5, x^2 - z^5, y^2 - z^3)$. (Hint: it may help to write $s = \frac{y}{z}$ when $z \neq 0$).

$W \subseteq V(x^3-y^5, x^2-z^5, y^2-z^3)$:

Let $(a,b,c) \in W$, then $(a,b,c) = (t^5, t^3, t^2)$ for some $t \in k$, then for $(x,y,z) = (a,b,c) = (t^5, t^3, t^2)$,

$x^3 - y^5 = (t^5)^3 - (t^3)^5 = 0$, $x^2 - z^5 = (t^5)^2 - (t^2)^5 = 0$, $y^2 - z^3 = (t^3)^2 - (t^2)^3 = 0$.

Thus $(a,b,c) \in V(x^3-y^5, x^2-z^5, y^2-z^3)$.

$V(x^3-y^5, x^2-z^5, y^2-z^3) \subseteq W$:

Let $(x_0, y_0, z_0) \in V(x^3-y^5, x^2-z^5, y^2-z^3)$, Then $x_0^3 - y_0^5 = 0$, $x_0^2 - z_0^5 = 0$, $y_0^2 - z_0^3 = 0$

If $z_0 = 0$, then $(x_0, y_0, z_0) = (0,0,0) = (0^5, 0^3, 0^2) \in W$.

If $z_0 \neq 0$, let $s = \frac{y_0}{z_0}$. Then $s^2 = \frac{y_0^2}{z_0^2} = \frac{z_0^3}{z_0^2} = z_0 \Rightarrow \underline{s^2 = z_0}$

$s^3 = \frac{y_0^3}{z_0^3} = \frac{y_0^3}{y_0^2} = y_0 \Rightarrow \underline{s^3 = y_0}$, $s^5 = \frac{y_0^5}{z_0^5} = \frac{x_0^3}{x_0^2} = x_0 \Rightarrow \underline{s^5 = x_0}$

Thus $(x_0, y_0, z_0) = (s^5, s^3, s^2) \in W \Rightarrow V(x^3-y^5, x^2-z^5, y^2-z^3) \subseteq W$.

Therefore, we get the equality.

b) Show that $xy - z^4 \in I(W) - \langle x^3 - y^5, x^2 - z^5, y^2 - z^3\rangle$.

Let $P \in W$, then $P = (t^5, t^3, t^2)$ for some $t \in k$, then $xy - z^4 = (t^5)\cdot(t^3) - (t^2)^4 = 0$

$xy - z^4$ vanishes at all $P \in W$, so $xy - z^4 \in I(W)$

If $xy - z^4 \in \langle x^3-y^5, x^2-z^5, y^2-z^5\rangle$ then

$xy - z^4 = h_1(x,y,z)\cdot(x^3-y^5) + h_2(x,y,z)\cdot(x^2-z^5) + h_3(x,y,z)\cdot(y^2-z^3)$ for some $h_1, h_2, h_3 \in k[x,y,z]$.

If $h_i = \sum_j c_{ij} x^{\alpha_{ij}} y^{\beta_{ij}} z^{\gamma_{ij}}$ is substituted, each term on the right side is divisible by one of x^3, y^5, x^2, z^5, y^2 or z^3, so there is no xy term on the right.

Thus $xy - z^4 \notin \langle x^3-y^5, x^2-z^5, y^2-z^3\rangle$

c) Show that if $f = x^4y^3z^2 + y^5z^3 + x^6$, then $f \notin \langle x^3 - y^5, x^2 - z^5, y^2 - z^3\rangle$.

$f(t^5, t^3, t^2) = (t^5)^4 \cdot (t^3)^3 \cdot (t^2)^2 + (t^3)^5 \cdot (t^2)^3 + (t^5)^6$

$= t^{33} + t^{21} + t^{30}$ → not 0 polynomial, hence not 0 function if k is infinite

Hence f does not vanish at every point of W if k is infinite

Thus, $f \notin I(W)$.

$f(t^5, t^3, t^2) = t^{33} + t^{21} + t^{30}$ cannot be 0 function:

Even if k is a finite field,

$f(1,1,1) = 3 = 0 \Rightarrow$ characteristic of k is 3 (For example $k = \mathbb{Z}_3$)

Then $f(-1) = 1 \neq 0$. So $f \notin I(W)$ for any field k.

Note that $W = V(x^3-y^5, x^2-z^5, y^2-z^3)$

$I = \langle x^3-y^5, x^2-z^5, y^2-z^3\rangle \subseteq I(W)$ We get $f \notin I(W) \Rightarrow \underline{f \notin I}$

5. (15 pts.) Let $S \subset k[x_1, ..., x_n]$ be an infinite set of polynomials (S can be countable or uncountable) and let $I = \langle S \rangle$ be the ideal generated by elements of S in $k[x_1, ..., x_n]$ (I consists of elements of the form $h_1 f_1 + h_2 f_2 + \cdots + h_r f_r$ where $f_i \in S$, $h_i \in k[x_1, ..., x_n]$ and $r \in \mathbb{N}$).
Show that there is a finite subset $\{g_1, g_2, ..., g_m\}$ of S such that $I = \langle g_1, g_2, ..., g_m \rangle$.

$I = \langle S \rangle$ is an ideal of $k[x_1, .., x_n]$, thus by Hilbert Basis Theorem I has a finite basis $\{F_1, F_2, .., F_r\}$, that is $I = \langle F_1, F_2, .., F_r \rangle$

For each $i = 1, 2, .., r$, $F_i \in I = \langle S \rangle$, so

$$F_i = h_{i1} f_{i1} + h_{i2} f_{i2} + \cdots + h_{iN_i} f_{iN_i} = \sum_{j=1}^{N_i} h_{ij} f_{ij} \quad \text{where } f_{ij} \in S,\ h_{ij} \in k[x_1, .., x_n]$$

so $F_i \in \langle f_{i1}, f_{i2}, f_{i3}, .., f_{iN_i} \rangle$.

Then for $S_i = \{f_{i1}, f_{i2}, .., f_{iN_i}\} \subseteq S$, define $\tilde{S} = S_1 \cup S_2 \cup \cdots \cup S_r$.
Each S_i is a finite set, S_i has N_i elements. $\tilde{S}$ is a finite subset of S.
$S_i \subseteq \tilde{S}$, so $\langle S_i \rangle \subseteq \langle \tilde{S} \rangle$, $F_i \in \langle S_i \rangle$, so $F_i \in \langle \tilde{S} \rangle$ for all $i = 1, 2, .., r$.
Thus $\{F_1, F_2, .., F_r\} \subseteq \langle \tilde{S} \rangle$
Then, $I = \langle F_1, F_2, .., F_r \rangle \subseteq \langle \tilde{S} \rangle \subseteq \langle S \rangle = I$
Therefore, $I = \langle F_1, .., F_r \rangle = \langle \tilde{S} \rangle = \langle S \rangle = I$, $\boxed{\langle \tilde{S} \rangle = I}$, $\tilde{S} \subseteq S$, $\tilde{S}$ is finite.

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or: • Verbally, each generator F_i is a finite linear combination of some $f_{ij} \in S$ with polynomial coefficients. The set of all these f_{ij} appearing in those linear combinations is a finite set $\tilde{S}$ (finitely many f_{ij} is used for each F_i, $i = 1, 2, .., r$), and $I = \langle \tilde{S} \rangle$ since $F_i \in \langle \tilde{S} \rangle$ for each i, and this implies $\langle F_1, .., F_r \rangle \subseteq \langle \tilde{S} \rangle$

F_i is a linear comb. of some elements of $\tilde{S}$ with polynomial coef.

$$\left.\begin{array}{l} I \subseteq \langle \tilde{S} \rangle \\ \tilde{S} \subseteq I \Rightarrow \langle \tilde{S} \rangle \subseteq I \end{array}\right\} \text{ so } I = \langle \tilde{S} \rangle$$

6. (2 × 4 pts.) Let $W = V(f_1, f_2)$, $Y = V(g_1, g_2)$ and $Z = V(h_1, h_2)$ be affine varieties in k^n where f_i, g_i and h_i are in $k[x_1, ..., x_n]$. In part (a) and (b) below, write a system of polynomial equations whose solution set is the given set. No explanation is necessary.

a) $W \cap (Y \cup Z)$

$Y \cup Z = V(g_1, g_2) \cup V(h_1, h_2) = V(g_1 h_1, g_1 h_2, g_2 h_1, g_2 h_2)$
$W \cap (Y \cup Z) = V(f_1, f_2) \cap V(g_1 h_1, g_1 h_2, g_2 h_1, g_2 h_2)$
$= V(f_1, f_2, g_1 h_1, g_1 h_2, g_2 h_1, g_2 h_2)$

System of polynomial equations: $f_1 = 0, f_2 = 0, g_1 h_1 = 0, g_1 h_2 = 0, g_2 h_1 = 0, g_2 h_2 = 0$.

b) $W \cup Y \cup Z$

$W \cup (Y \cup Z) = V(f_1, f_2) \cup V(g_1 h_1, g_1 h_2, g_2 h_1, g_2 h_2)$
$= V(f_1 \cdot g_1 h_1, f_1 \cdot g_1 h_2, f_1 \cdot g_2 h_1, f_1 \cdot g_2 h_2, f_2 \cdot g_1 h_1, f_2 \cdot g_1 h_2, f_2 g_2 h_1, f_2 g_2 h_2)$

System of equations: $f_a \cdot g_b \cdot h_c = 0$ for all (a, b, c), $a, b, c \in \{1, 2\}$
(8 equations)