M E T U
Department of Mathematics

	Ideals, Varieties and Algorithms **Midterm 2**	

Code : *Math 473* Acad. Year : *2019-2020* Semester : *Fall* Instructor : *Tolga Karayayla* Date : *10.12.2019* Time : *17.40* Duration : *120 minutes*	Last Name : First Name : Student ID : Department : Signature : **5** Questions on **4** Pages SHOW DETAILED WORK!

1	2	3	4	5		

NOTE: k is a field in all questions below.

1. (3×5 pts.) For a fixed monomial order let G be a Groebner basis for an ideal $I \subset k[x_1, ..., x_n]$, and let $\overline{f}^G$ denote the unique remainder of $f \in k[x_1, ..., x_n]$ on division by elements of G.

a) Show that $\overline{f}^G = \overline{g}^G$ if and only if $f - g \in I$.

b) Conclude that $\overline{f+g}^G = \overline{f}^G + \overline{g}^G$.

c) Conclude that $\overline{f \cdot g}^G = \overline{\overline{f}^G \cdot \overline{g}^G}^G$.

2. (20 + 5 + 5 pts.) a) Use Buchberger's Algorithm to compute a Groebner basis of the ideal $I = \langle x^2y^2 + x + 1, xy^3 + y + 1\rangle$ with respect to *lex* order where $x > y > z$.

2.b) What is the reduced Groebner basis of I?

c) How many points are there in $V(I) \subset \mathbb{C}^2$?

3. (15 + 5 pts.) a) Is $\{x^3z + xy, xy^4 - xz\}$ a Groebner basis of the ideal $I = \langle x^3z + xy, xy^4 - xz \rangle$ with respect to *grlex* order where $x > y > z$?

b) Is $x^2y^3z^2 + xy^4z$ in I?

4. (6 + 4 + 10 pts.) Let $x = st$, $y = s^2$, $z = t^2$ be a parametrization. A Groebner basis for the ideal $I = \langle x - st, y - s^2, z - t^2 \rangle \subset \mathbb{C}[s, t, x, y, z]$ for *lex* order with $s > t > x > y > z$ is given as $G = \{st - x, s^2 - y, t^2 - z, x^2 - yz, sx - ty, sz - tx\}$.

a) Write down generators of the elimination ideals I_1 and I_2.

b) What is the smallest affine variety in $\mathbb{C}^3$ containing the image of this parametrization.

c) Show that the image of the parametrization equals the variety in part (b) in $\mathbb{C}^3$, but if the parametrization is considered from $\mathbb{R}^2$ to $\mathbb{R}^3$, then the image of the parametrization is not equal to this variety in $\mathbb{R}^3$.

5. (15 pts.) Let $I = \langle f_1, f_2, ..., f_r \rangle \subset k[x_1, x_2, ..., x_n]$ be a given ideal. How can you determine whether I is a principal ideal or not? Describe an algorithm for this problem. DO NOT write down a code for the algorithm, only describe the process step by step in words. Explain why this algorithm gives the result.