M E T U
Department of Mathematics

<table>
<tr><td></td><td colspan="2">Ideals, Varieties and Algorithms
Midterm 1</td><td></td></tr>
<tr><td colspan="2">Code : Math 473
Acad. Year : 2019-2020
Semester : Fall
Instructor : Tolga Karayayla

Date : 07.11.2019
Time : 17.40
Duration : 120 minutes</td><td colspan="2">Last Name :
First Name : Student ID :
Department :
Signature :

6 Questions on 4 Pages
SHOW DETAILED WORK!</td></tr>
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NOTE: k is a field in all questions below.

1. (15 pts.) Use *lex* order with $x > y$ and perform the Division Algorithm to divide $f = x^2y + 2y^3 + 3x^4y^5$ by (f_1, f_2, f_3) in $\mathbb{Q}[x, y]$ where $f_1 = x^3y^2 + x + 2$, $f_2 = xy^3 + 2y$ and $f_3 = y^2 + 3$.

2. (15 pts.) Fact: $\{g_1, g_2, ..., g_r\} \subset I$ is a Groebner basis of an ideal $I \subset k[x_1, x_2, ..., x_n]$ if and only if for every $f \in I$ there is at least one g_i such that $LT(g_i)$ divides $LT(f)$.
Use this fact to show that $\{g_1, g_2, g_3\}$ is not a Groebner basis of $I = \langle g_1, g_2, g_3 \rangle \subset \mathbb{R}[x, y, z]$ in *grlex* order with $x > y > z$ where $g_1 = x^2y + y^2z^2 + z$, $g_2 = x^4 + 2xy^4z + 1$ and $g_3 = x^2y^2 + yz + 2$.

3. (10+10 pts.) a) For any monomial order $>$ show that $LT(f \cdot g) = LT(f) \cdot LT(g)$ for any nonzero polynomials f and g in $k[x_1, x_2, ..., x_n]$.

b) Let $I = \langle g \rangle \subset k[x_1, ..., x_n]$ where $g \neq 0$ be a principal ideal. Show that if G is a finite subset of I containing cg for some $c \in k - \{0\}$, then G is a Groebner basis of I.

4. (3×9 pts.) a) Let $W = \{(t^5, t^3, t^2) \in k^3 | t \in k\}$. Show that $W = V(x^3 - y^5, x^2 - z^5, y^2 - z^3)$. (Hint: it may help to write $s = \dfrac{y}{z}$ when $z \neq 0$).

b) Show that $xy - z^4 \in I(W) - \langle x^3 - y^5, x^2 - z^5, y^2 - z^3 \rangle$.

c) Show that if $f = x^4y^3z^2 + y^5z^3 + x^6$, then $f \notin \langle x^3 - y^5, x^2 - z^5, y^2 - z^3 \rangle$.

5. (15 pts.) Let $S \subset k[x_1, ..., x_n]$ be an infinite set of polynomials (S can be countable or uncountable) and let $I = \langle S \rangle$ be the ideal generated by elements of S in $k[x_1, ..., x_n]$ (I consists of elements of the form $h_1 f_1 + h_2 f_2 + \cdots + h_r f_r$ where $f_i \in S$, $h_i \in k[x_1, ..., x_n]$ and $r \in \mathbb{N}$).
Show that there is a finite subset $\{g_1, g_2, ..., g_m\}$ of S such that $I = \langle g_1, g_2, ..., g_m \rangle$.

6. (2×4 pts.) Let $W = V(f_1, f_2)$, $Y = V(g_1, g_2)$ and $Z = V(h_1, h_2)$ be affine varieties in k^n where f_i, g_i and h_i are in $k[x_1, ..., x_n]$. In part (a) and (b) below, write a system of polynomial equations whose solution set is the given set. No explanation is necessary.

a) $W \cap (Y \cup Z)$

b) $W \cup Y \cup Z$