M E T U
Department of Mathematics

	Ideals, Varieties and Algorithms **Final**	

Code : *Math 473* Acad. Year : *2019-2020* Semester : *Fall* Instructor : *Tolga Karayayla* Date : *13.01.2020* Time : *9:30* Duration : *120 minutes*	Last Name : First Name : Student ID : Department : Signature : **6** Questions on **4** Pages SHOW DETAILED WORK!

1	2	3	4	5	6		

NOTE: k is a field in all questions below.

1. (3×7 pts.) a) Show that $f \in \mathbb{C}[x]$ has a multiple root (a root with multiplicity greater than 1, a repeated root) if and only if $\mathrm{Res}(f, f', x) = 0$.

b) Let $f(x,y) = x^3y + x^2y^2 + x^2 + xy + 2y + 1$ and $g(x,y) = x^2y + 2x + y + 1$ in $k[x,y]$. Write down $\mathrm{Res}(f,g,x)$ and $\mathrm{Res}(f,g,y)$ as determinants (do not expand/calculate the determinants).

c) When two polynomials $p(x,y)$ and $q(x,y) \in k[x,y]$ are given, can you determine whether p and q are relatively prime or not by using resultants without factorizing p and q and without calculating their greatest common divisor? If yes, how and why? If no, why not?

2. (2×7 pts.) a) Let k be an infinite field. Let $f, g \in k[x_1, x_2, ..., x_n]$ such that f is a nonzero polynomial satisfy $g(x_1, ..., x_n) = 0$ for all $(x_1, ..., x_n) \in k^n - V(f)$. Show that g is the zero polynomial. (Hint: consider fg).

b) Let k be an infinite field and $W \subset k^n$ where $W \neq k^n$ be an affine variety. Show that if $g \in k[x_1, ..., x_n]$ vanishes on all points of $k^n - W$, then g is the zero polynomial (you can use the fact given in part (a) above even if you did not answer part (a)).

3. (13 pts.) For $J = \langle xy, (x-y)x \rangle \subset k[x, y]$, show that $\sqrt{J} = \langle x \rangle$. (Note that k here can be any field, k is not necessarily algebraically closed.)

4. (2 × 8 pts.) a) Let $J = \langle x^2 + y^2 - 1, y - 1 \rangle \subset k[x, y]$. Find an $f \in I(V(J))$ such that $f \notin J$.

b) If k is $\mathbb{C}$, is J a radical ideal?

5. (12 pts.) For the rational parametrization $x = \dfrac{st^2}{1 + s^2t^3}$, $y = \dfrac{s + t}{st}$, $z = \dfrac{s^2 + t^2}{st + t^4}$ in $\mathbb{R}^3$, how can you find the smallest affine variety in $\mathbb{R}^3$ which contains the image of this parametrization? Explain the process step by step, do not carry out the computations.

6. (3×8 pts.) For each problem below, explain its solution method/procedure step by step. Do not prove why this procedure solves the problem, only list what to do in order to solve the problem.

a) When an ideal $I = \langle f_1, f_2, ..., f_s \rangle$ and a polynomial f is given in $k[x_1, ..., x_n]$, how do you determine whether $f \in \sqrt{I}$ or not?

b) For two ideals $I = \langle f_1, f_2, f_3 \rangle$ and $J = \langle g_1, g_2 \rangle$ in $k[x, y]$, how do you find generators of the ideal $I \cap J$?

c) For two polynomials $f, g \in k[x_1, x_2, ..., x_n]$, how do you calculate a greatest common divisor of f and g without using factorization of f and g into a product of irreducibles? (Note that $n \geq 2$ here. Even if you did not answer part (b) above, here you can use part (b) directly without explaining how it is solved)