15.1 | Vector Fields

Vector fields in $\mathbb{R}^2$:

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$
$$= \langle P(x,y), Q(x,y)\rangle$$
$$= (P(x,y), Q(x,y))$$

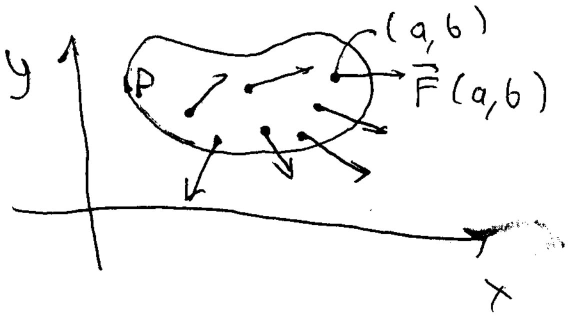
$\vec{F}: D \to \mathbb{R}^2$ where $D \subseteq \mathbb{R}^2$, D: domain of the vector field $\vec{F}$

For every $(x,y) \in D$, $\vec{F}(x,y)$ is a vector in $\mathbb{R}^2$ which depends on (x,y)

$\vec{F}(x,y)$ is considered as a vector whose initial point is $(x,y) \in D$

$P(x,y)$: x-component of $\vec{F}(x,y)$
$Q(x,y)$: y component of $\vec{F}(x,y)$

$\left.\right\}$ P and Q are component functions of the vector field $\vec{F}$.



Vector Fields in $\mathbb{R}^3$:

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\cdot\vec{j} + R(x,y,z)\cdot\vec{k}$$
$$= \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

$\vec{F}: D \to \mathbb{R}^3$ where $D \subseteq \mathbb{R}^3$, D: domain of $\vec{F}$.

$P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$: component functions of $\vec{F}$.

- We'll consider vector fields $\vec{F}$ with component functions which have continuous first order partial derivatives.

16.1 | Gradient, Divergence and Curl

1) Gradient of a function f.

For $f(x,y)$,

$$\vec{\nabla} f(x,y) = (f_x(x,y), f_y(x,y)) = f_x(x,y)\cdot\vec{i} + f_y(x,y)\cdot\vec{j}$$

↳ Gradient of f

For $f(x,y,z)$:

$$\vec{\nabla} f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$$
$$= f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\cdot\vec{k}$$

For a function f, $\vec{\nabla f}$ is a vector field.

Notation: $\vec{\nabla} f(x,y,z) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

2) Divergence of a vector field $\vec{F}$.

For $\vec{F}(x,y) = (P(x,y), Q(x,y))$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = P_x(x,y) + Q_y(x,y)$$
$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

For $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = Q_x(x,y,z) + P_y(x,y,z) + R_z(x,y,z)$$
$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (P, Q, R) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

For a vector field $\vec{F}$, divergence of $\vec{F}$ ($\operatorname{div}\vec{F}$) is a function (a scalar function)

3) Curl of a vector field $\vec{F}$ in $\mathbb{R}^3$

For $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$,

curl of $\vec{F}$ is defined as

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = (R_y - Q_z, -R_x + P_z, Q_x - P_y)$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (P, Q, R)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$\nabla \times \vec{F}$ is a vector field in $\mathbb{R}^3$

- Gradient of a function will play an important role in conservative vector fields and path independent line integrals

- Divergence of a vector field appears in the "Divergence Theorem" which relates surface integrals on boundary surfaces to triple integrals (not content of this course)

- Curl of a vector field appears in "Stokes' Theorem" which relates a line integral of a vector field along a boundary curve to a surface integral (not content of this course), and also appears in "Green's Theorem" which relates line integral of a vector field along a simple closed plane curve to a double integral on the enclosed region by this curve. (the last subject of the course).

15.2 | Conservative Vector Fields

A vector field $\vec{F}$ is called conservative on D if there is a function f such that $\vec{F} = \vec{\nabla f}$ on D.

For $\vec{F}(x,y) = (P(x,y), Q(x,y))$,

$\vec{F}(x,y)$ is conservative on $D \subseteq \mathbb{R}^2 \iff \vec{F}(x,y) = \vec{\nabla f}(x,y)$ for all $(x,y) \in D$ for some function $f(x,y)$ defined on D

$\iff (P(x,y), Q(x,y)) = (f_x(x,y), f_y(x,y))$ for all $(x,y) \in D$.

For $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$,
$\vec{F}(x,y,z)$ is conservative on $D \subseteq \mathbb{R}^3$

$\iff \vec{F}(x,y,z) = \vec{\nabla f}(x,y,z)$ for all $(x,y,z) \in D$ for some function $f(x,y,z)$ defined on D.

$\iff (P, Q, R) = (f_x, f_y, f_z)$ on D

$$\iff \left.\begin{aligned} P(x,y,z) &= f_x(x,y,z) \\ Q(x,y,z) &= f_y(x,y,z) \\ R(x,y,z) &= f_z(x,y,z) \end{aligned}\right\} \text{ for all } (x,y,z) \in D \text{ for some function } f(x,y,z) \text{ defined on D.}$$

Potential function,

If $\vec{F}$ is a conservative vector field on D and $\vec{F} = \vec{\nabla f}$ on D, then f is called a potential function of $\vec{F}$ on D.

Example:

Show that $\vec{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} + 1 \right)$ is a conservative vector field on $\mathbb{R}^2 - \{(0,0)\}$

Solution: Let $D = \mathbb{R}^2 - \{(0,0)\} \subseteq \mathbb{R}^2$.

We'll show that $\vec{F}$ is conservative on D by finding a potential function $f(x,y)$ for $\vec{F}$, that is, by showing the existence of $f(x,y)$ such that

$\vec{F}(x,y) = \vec{\nabla} f$ on $D = \mathbb{R}^2 - \{(0,0)\}$.

$$\vec{F}(x,y) = \vec{\nabla} f(x,y) \iff \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} + 1 \right) = \left(f_x(x,y), f_y(x,y) \right)$$

$$\iff f_x(x,y) = \frac{x}{x^2+y^2} \wedge f_y(x,y) = \frac{y}{x^2+y^2} + 1.$$

$$f_x(x,y) = \frac{x}{x^2+y^2} \Rightarrow f(x,y) = \int \frac{x}{x^2+y^2}\, dx = \frac{1}{2} \ln|x^2+y^2| + \underbrace{C(y)}_{\text{A function of } y}$$

$$f_y(x,y) = \frac{y}{x^2+y^2} + 1$$

$$\frac{\partial}{\partial y}\left(\frac{1}{2} \ln|x^2+y^2| + C(y) \right) = \frac{y}{x^2+y^2} + 1$$

(Since integration is w.r.t. x, y is treated as constant, the arbitrary constant after integration is a function of y, $C(y)$).

$$\Rightarrow \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2y + C'(y) = \frac{y}{x^2+y^2} + 1 \Rightarrow C'(y) = 1$$

$$\Rightarrow C(y) = \int 1\, dy = y + E$$

Therefore,

$f(x,y) = \frac{1}{2} \ln|x^2+y^2| + y + E$ where E is a constant.

for some $E \in \mathbb{R} \Rightarrow \vec{F}(x,y) = \vec{\nabla} f(x,y)$ on D, hence $\vec{F}$ is conservative on D.

Theorem,

1) For $\vec{F}(x,y)=(P(x,y),Q(x,y))$ such that P and Q have continuous first order partial derivatives on $D\subseteq\mathbb{R}^2$,

$\vec{F}$ is conservative on $D \Rightarrow Q_x(x,y)=P_y(x,y)$ for all $(x,y)\in D$.

2) For $\vec{F}(x,y,z)=(P(x,y,z),Q(x,y,z),R(x,y,z))$ such that P,Q,R have continuous first order partial derivatives on $D\subseteq\mathbb{R}^3$,

$\vec{F}$ is conservative on $D \Rightarrow \left.\begin{array}{l} Q_x=P_y \\ Q_z=R_y \\ R_x=P_z \end{array}\right\}$ on D (In short $\underbrace{\nabla\times\vec{F}}_{\text{curl of } \vec{F}}=\vec{0}$ on D)

Proof: Proof of 2 is as follows, proof of 1 is similar.

Let $\vec{F}=\vec{\nabla} f$ on $D\subseteq\mathbb{R}^3$, then

$(P,Q,R)=(f_x,f_y,f_z)$

$P=f_x,\ Q=f_y,\ R=f_z$ on D.

First order partials of P,Q,R are second order partials of f and are continuous on D (as assumed in the statement of the theorem)

By equality of mixed partials, we get

$$Q_x=(f_y)_x=f_{yx}=f_{xy}=(f_x)_y=P_y \Rightarrow Q_x=P_y$$

$$Q_z=f_{yz}=f_{zy}=R_y \Rightarrow Q_z=R_y$$

$$R_x=f_{zx}=f_{xz}=P_z \Rightarrow R_x=P_z$$

Then $\nabla\times\vec{F}=(R_y-Q_z,\ P_z-R_x,\ Q_x-P_y)=(0,0,0)=\vec{0}$ on D.

Example

Show that $\vec{F}(x,y,z) = (2xy,\ x^2+2yz+1,\ y^2+x)$ is not a conservative vector field on any domain $D \subseteq \mathbb{R}^3$.

Solution

$\vec{F}(x,y,z) = (P,Q,R)$ where $P = 2xy$, $Q = y^2+2yz+1$, $R = y^2+x$

For any domain $D \subseteq \mathbb{R}^3$, P,Q,R have continuous first order partials on D. Then by the previous theorem, we have

$$\vec{F} \text{ is conservative on } D \Rightarrow \begin{array}{l} Q_x = P_y \\ R_y = Q_z \\ R_x = P_z \end{array} \text{ on } D.$$

We check these 3 equations:

$Q_x = 2x = P_y \Rightarrow Q_x = P_y$ holds

$R_y = 2y = Q_z \Rightarrow R_y = Q_z$ holds, But $R_x = 1 \neq 0 = P_z$

$R_x = P_z$ does not hold

Thus $\vec{F}$ is not conservative on D.

Second Solution

$$\vec{F}(x,y,z) = \nabla f(x,y,z) \Leftrightarrow \begin{array}{l} P = f_x \\ Q = f_y \\ R = f_z \end{array}$$

$f_x(x,y,z) = P \Rightarrow f(x,y,z) = \int P\, dx = \int 2xy\, dx = x^2y + C(y,z)$

$f_y = Q \Rightarrow \dfrac{\partial}{\partial y}\left(x^2y + C(y,z)\right) = x^2 + 2yz + 1$

$x^2 + C_y(y,z) = x^2 + 2yz + 1 \Rightarrow C_y(y,z) = 2yz + 1$

Then $C(y,z) = \int 2yz + 1\, dy = y^2z + y + E(z)$ — We didn't write $E(x,z)$ since C is a func. of y and z. variable x is not involved in this integration w.r.t. y)

Then $f(x,y,z) = x^2y + C(y,z)$

$= x^2y + y^2z + y + E(z)$

Then

$f_z = R \Rightarrow \dfrac{\partial}{\partial z}\left(x^2y + y^2z + y + E(z)\right) = y^2 + x$

$0 + y^2 + 0 + E'(z) = y^2 + x \Rightarrow E'(z) = x$; contradiction, $E'(z)$ cannot depend on x!!

Due to this contradiction, no such potential function f exists.

Theorem

1) Let $\vec{F}(x,y) = (P(x,y), Q(x,y))$ where P and Q have continuous first order partial derivatives on $D \subseteq \mathbb{R}^2$ and D is simply connected. Then
$\vec{F}$ is conservative on $D \iff Q_x = P_y$ on D.

2) Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$
where P, Q and R have continuous first order partials on $D \subseteq \mathbb{R}^3$ and D is simply connected. Then

$$\vec{F} \text{ is conservative on } D \iff \begin{matrix} Q_x = P_y \\ Q_z = R_y \\ P_z = R_x \end{matrix} \quad \text{on } D.$$

Proof: Omitted. Proof of 1 uses Green's Thm, proof of 2 uses Stokes' Thm

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Simply Connected Domains:

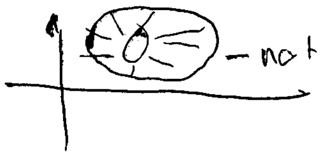
A subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$ is called simply connected if every closed curve C in D can be shrunk to a point in D by continuously deforming the curve C by always staying in D (while deforming C, cutting and pasting is not allowed, but the curve can cross itself)

Examples

$\mathbb{R}^2$ is simply connected

– first octant is simply connected.

$\mathbb{R}^2 - \{(0,0)\}$ is not simply connected

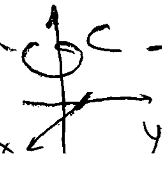


– not simply connected

* $D \subseteq \mathbb{R}^2$ is simply connected if and only if D has no hole in it.

$\mathbb{R}^3$ and $\mathbb{R}^3 - \{(0,0,0)\}$ are simply connected.

$\mathbb{R}^3 - \{z\text{-axis}\}$ is not simply connected



– cannot shrink to a point without passing through z axis

we can't shrink C to a point by staying in $\mathbb{R}^3$ – z axis