A Fourier–Bessel Expansion for Solving Radial Schrödinger Equation in Two Dimensions

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ABSTRACT

The spectrum of the two-dimensional Schrödinger equation for polynomial oscillators bounded by infinitely high potentials, where the eigenvalue problem is defined on a finite interval \( r \in [0, L] \), is variationally studied. The wave function is expanded into a Fourier–Bessel series, and matrix elements in terms of integrals involving Bessel functions are evaluated analytically. Numerical results presented accurate to 30 digits show that, by the time \( L \) approaches a critical value, the low-lying state energies behave almost as if the potentials were unbounded. The method is applicable to multiwell oscillators as well.

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1. Introduction

In his recent articles [1–3], Taşeli modified the usual requirement that the wave function should tend to zero at infinity and showed that the eigensolution of the Dirichlet boundary value problem can be effectively used to find the spectrum of an unbounded problem in one dimension. In these works, the eigenfunctions satisfying the boundary value problem

\[
\frac{d^2 \Psi}{dx^2} + \mu^2 \Psi = 0, \quad \Psi(a) = \Psi(b) = 0 \quad (1.1)
\]

were employed as the basis set in the Rayleigh–Ritz variational method. The approach has a natural extension to the two-dimensional Schrödinger equation written in Cartesian coordinates. Actually, two-dimensional anharmonic oscillators can be treated in a similar fashion by means of the boundary value problem defined by

\[
\frac{d^2 \Psi}{dx^2} + \frac{d^2 \Psi}{dy^2} + \mu^2 \Psi = 0, \quad \Psi|_\Gamma = 0, \quad (1.2)
\]

where \( \Gamma \) denotes the boundary of a finite rectangular region in the \( xy \)-plane [4].

In this study, we examined the dimensionless radial Schrödinger equation in the cylindrical polar...
coordinates:
\[
\left[-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{l^2}{r^2} + V(r)\right] \Psi(r) = E \Psi(r),
\]
\[r \in [0, \infty), \quad (1.3)\]
where \( l = 0, 1, \ldots, V(r) \) and \( E \) stand for the magnetic quantum number, the potential function, and the energy eigenvalue, respectively. It is obvious that the coupling might depend on the direction. We have, however, omitted the angle dependence of the potential for the sake of dealing with a system which can be investigated via an ordinary differential equation. Therefore, \( l \) characterizes the angular dependence of the system in a global sense, and the wave function \( \Psi \) has been regarded as a function of the single variable \( r \).

The accompanying boundary conditions of (1.3) are the regularity and the appropriately vanishing behavior of the wave function specified as \( r \to 0 \) and \( r \to \infty \), respectively. The regularity condition implies that
\[
\Psi(r) = O(r^l) \quad (1.4)
\]
as \( r \to 0 \). The second condition, however, is replaced by
\[
\Psi(L) = 0 \quad (1.5)
\]
when the interval is truncated to \([0, L]\). Such a truncation is clearly motivated by the success of the simple technique presented in the aforementioned articles [1–4]. The question which now arises is whether there exists a corresponding basis set, preferably in terms of elementary or special functions of mathematical analysis for solving (1.3). Fortunately, in the case of the radial Schrödinger equation, the Bessel functions of the first kind are to be shown in Section 2 to play the same role with the trigonometric basis.

The potential function \( V(r) \) in (1.3) is taken as a general polynomial:
\[
V(r) = \sum_{j=1}^{M} v_{2j} r^{2j}, \quad v_{2M} > 0, \quad M = 1, 2, \ldots \quad (1.6)
\]
in \( r^2 \). The positiveness of the dominant coupling constant \( v_{2M} \) is sufficient to make the potential bounded below. Therefore, the operator being considered has now a purely discrete spectrum. The special case of the harmonic oscillator, \( V(r) = r^2 \), admits exact solutions in the unbounded domain of \( r \) of the form
\[
\Phi_n(r) = e^{-(1/2)r^2} L_n(r^2), \quad E_n, l = 2(2n + l + 1),
\]
\[n, l = 0, 1, \ldots, \quad (1.7)\]
where \( L_n(r) \) denotes the associated Laguerre polynomials.

In general, the asymptotic behavior of the wave function as \( r \to \infty \) completely depends on the dominant coupling. As a result, it is rather difficult to introduce a trial function reflecting the desired properties of the solution for an arbitrary anharmonic interaction [5]. The definition of the problem in a finite interval \( r \in [0, L] \), however, makes it possible to consider a general polynomial potential rather than a specific one. The idea is based upon regarding the boundary value \( L \) as a nonlinear optimization parameter to be determined in such a way that the spectrum fits to the spectrum of the corresponding unbounded problem, where \( L \to \infty \), to any prescribed accuracy. Moreover, a model of this kind, namely, an enclosed quantum mechanical system, is of importance not only for finding the spectrum of an unbounded one but also its various applications in several fields [6] (and the references cited therein).

Within these perspectives, Section 2 sets out the basic variational formulation of the problem. Section 3 includes the evaluation of integrals containing Bessel functions. The last section presents the applications of the method and concludes the article with a discussion of the results.

### 2. The Fourier–Bessel Expansion

In this section, we begin with solving the unperturbed Schrödinger equation defined by
\[
\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \mu^2 - \frac{l^2}{r^2}\right) \Phi(r) = 0, \quad r \in [0, L],
\]
\[(2.1)\]
subject to
\[
\Phi(r) = O(r^l) \quad \text{as} \quad r \to 0, \quad \Phi(L) = 0, \quad (2.2)
\]
where \( \mu \) is a constant, and we have assumed a potential function of the form
\[
V(r) = \begin{cases} 0, & 0 \leq r < L, \\ \infty, & r > L. \end{cases} \quad (2.3)
\]
It is readily shown that the two linearly independent solutions of (2.1) are \( j_l(\mu r) \) and \( Y_l(\mu r) \), namely, the Bessel functions of the first and the second kind, respectively. However, the Bessel functions of the second kind do not remain finite at \( r = 0 \) so that we may take

\[
\Phi(r) = cj_l(\mu r), \quad (2.4)
\]

which behaves correctly at the origin, where \( c \) is some normalization constant. More specifically,

\[
\Phi(r) \approx \frac{c\mu^l}{2^l l!} r^l \quad (2.5)
\]
as \( r \to 0 \). Imposing the second condition, we see that (2.4) is the required solution if \( \mu \) is a positive root of the equation

\[
j_l(\mu L) = 0. \quad (2.6)
\]

We know from the theory of Bessel functions that there is an enumerable infinite set of roots \( \mu = \mu_1, \mu_2, \ldots, \mu_n, \ldots \) satisfying (2.6) [7]. Furthermore, when \( c \) is properly chosen, the sequence of functions

\[
\Phi_n(r) = \frac{\sqrt{2}}{Lj_{l+1}(\mu_n L)} j_l(\mu_n r), \quad n = 1, 2, \ldots
\]

forms an orthonormal set over the range of \( r, \quad r \in [0, L] \), with respect to the weighting function \( r \). Thus, we have

\[
\int_0^L r\Phi_n(r)\Phi_m(r) \, dr = \delta_{nm} \quad (2.8)
\]

for all values of \( n \) and \( m \).

The orthonormality property suggests the expansion of the wave function in the form

\[
\Psi(r) = \sum_{n=1}^{\infty} a_n \Phi_n(r) = \frac{\sqrt{2}}{L} \sum_{n=1}^{\infty} \frac{a_n}{j_{l+1}(\mu_n L)} j_l(\mu_n r), \quad (2.9)
\]

which is called the Fourier–Bessel expansion of \( \Psi(r) \). The theory of such an expansion is given by Watson in [8]. Substituting (2.9) into (1.3), multiplying the result by \( r\Phi_m(r) \), and integrating from zero to \( L \), we obtain

\[
\sum_{n=1}^{\infty} \left[ \mu_n^2 \delta_{nm} + \int_0^L V(r) r\Phi_n(r)\Phi_m(r) \, dr - E\delta_{nm} \right] a_n = 0, \quad (2.10)
\]

for \( m = 1, 2, \ldots \). If we define the variational matrix to be

\[
H_{nm} = \mu_n^2 \delta_{nm} + \int_0^L V(r) r\Phi_n(r)\Phi_m(r) \, dr, \quad (2.11)
\]

we then arrive at the secular equations of the form

\[
\sum_{n=1}^{\infty} (H_{nm} - E\delta_{nm}) a_n = 0, \quad m = 1, 2, \ldots, \quad (2.12)
\]

for determination of the coefficients \( a_n \). Using (1.6) and making a simple change of variable, the variational matrix is expressible as

\[
H_{nm} = (\alpha_n/L)^2 \delta_{nm} + \sum_{j=1}^{M} v_{2j} L^2 l_{nm}^{2j}, \quad (2.13)
\]

wherein \( l_{nm}^{2j} \) stands for the integral

\[
I_{nm}^{2j} = c_{nm}^{(2j)} \int_0^1 \xi^{2j+1} j_l(\alpha_n \xi) j_l(\alpha_m \xi) \, d\xi \quad (2.14)
\]
to be determined for each \( l \). Hereafter, \( \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots \) denote the positive zeros of \( j_l(x) \), and the constant \( c_{nm}^{(2j)} \) is given by

\[
c_{nm}^{(2j)} = \frac{2}{I_{l+1}^{2j+1}(\alpha_n) I_{l+1}^{2j+1}(\alpha_m)}. \quad (2.15)
\]

It is apparent that \( I_{nm}^{2j} \) is symmetric in \( n \) and \( m \) and, hence, that \( H_{nm} \) is symmetric.

In the coming section, we shall show that the integrals in (2.14) may be evaluated recursively. In particular,

\[
I_{nm}^{(l,0)} = \delta_{nm} \quad (2.16)
\]

provided that \( j_l(\alpha_n) = 0 \) for any \( l \) fixed.

3. Evaluation of Matrix Elements

In this section, the integrals appearing in the matrix elements are evaluated analytically for \( l = 0 \)}
and $l = 1$, which are representative for the other cases. Since such integrals are not so trivial to deal with, we feel that we should give a sketch of the derivation steps. So let us consider (2.14) setting $l = 0$,

$$I_{nm}^{(0,j)} = c_{nm}^{(0)} \int_{0}^{1} \xi^{2j+1} J_{0}(\alpha_{n} \xi) J_{0}(\alpha_{m} \xi) \, d\xi;$$

$$n, m = 1, 2, \ldots, \quad (3.1)$$

with $J_{0}(\alpha_{n}) = 0$ and $j = 1, 2, \ldots, M$. Integrating by parts and using some well-known properties of the Bessel functions, we see that (3.1) may be written as

$$I_{nm}^{(0,j)} = \frac{2j}{\alpha_{n}^{2} - \alpha_{m}^{2}} (\alpha_{m} A_{mn} - \alpha_{n} A_{nm}), \quad (3.2)$$

when $n \neq m$, where

$$A_{pq} = c_{pq}^{(0)} \int_{0}^{1} \xi^{2j+1} J_{1}(\alpha_{p} \xi) J_{1}(\alpha_{q} \xi) \, d\xi \quad (3.3)$$

and $A_{pq} \neq A_{q,p}$. Integration by parts of the last integral leads to

$$\alpha_{m} A_{nm} + \alpha_{n} A_{mn} = 2 - 2(j - 1) P_{nm}^{(l,j-1)}, \quad \text{for } j > 0, \quad (3.4)$$

where a temporary quantity $P_{nm}^{(1,j)}$,

$$P_{nm}^{(1,j)} = c_{nm}^{(0)} \int_{0}^{1} \xi^{2j+1} J_{1}(\alpha_{n} \xi) J_{1}(\alpha_{m} \xi) \, d\xi;$$

$$P_{nm}^{(1,j)} = P_{nm}^{(1,j-1)}, \quad (3.5)$$

has been introduced. It should be noticed that $P_{nm}^{(1,j)}$ is not a constant multiple of $I_{nm}^{(1,j)}$ defined by (2.14) since $\alpha_{n}$’s are not the zeros of $J_{1}(x)$ in this case of $l = 0$. If we now make use of the identity

$$\frac{d}{d\xi} \left[ \xi^{2j} J_{0}(\alpha_{n} \xi) J_{0}(\alpha_{m} \xi) \right]$$

$$= 2j \xi^{2j-1} J_{0}(\alpha_{n} \xi) J_{0}(\alpha_{m} \xi) - \alpha_{n} \xi^{2j} J_{1}(\alpha_{n} \xi) J_{0}(\alpha_{m} \xi) - \alpha_{m} \xi^{2j} J_{0}(\alpha_{n} \xi) J_{1}(\alpha_{m} \xi); \quad (3.6)$$

it follows immediately that

$$\alpha_{n} A_{nm} + \alpha_{m} A_{mn} = 2j I_{nm}^{(0,j-1)}, \quad \text{for } j > 0, \quad (3.7)$$

which is independent of (3.4). Solving simultaneously (3.4) and (3.7) for $A_{nn}$ and $A_{nn}$ and substituting into (3.2), we find that

$$I_{nm}^{(0,j)} = \frac{4j}{(\alpha_{n}^{2} - \alpha_{m}^{2})^{2}} \left[ 2\alpha_{n} A_{mn} \left[ 1 - (j - 1) P_{nm}^{(l,j-1)} \right] \right. \left. - 2 \alpha_{n} \alpha_{m} I_{nm}^{(0,j-1)} \right] \quad (3.8)$$

for $j > 0$ and $m \neq n$. Likewise, we obtain the relation

$$P_{nm}^{(1,j)} = \frac{4j}{(\alpha_{n}^{2} - \alpha_{m}^{2})^{2}} \times \left\{ (\alpha_{n}^{2} + \alpha_{m}^{2}) \left[ 1 - (j - 1) P_{nm}^{(l,j-1)} \right] \right. \left. - 2 \alpha_{n} \alpha_{m} I_{nm}^{(0,j-1)} \right\} \quad (3.9)$$

for the calculation of $P_{nm}^{(1,j)}$ recursively. The initial conditions for these recursions are

$$I_{nm}^{(0,0)} = 0 \quad (3.10)$$

and

$$P_{nm}^{(1,0)} = 0, \quad (3.11)$$

respectively. The first condition is a consequence of the orthogonality of the Bessel functions, and the second one can be deduced from the integral (3.5) with $j = 0$.

Whenever $m = n$, $I_{nn}^{(0,j)}$ may be derived by means of certain limit operations as $\alpha_{n} \to \alpha_{n}$ using l’Hospital rule. It is, however, easier to follow an alternative way. Indeed, noting that (3.4) and (3.7) are valid for $m = n$ as well, we show that

$$(j + 1) I_{nn}^{(0,j)} = 1 - j P_{nn}^{(1,j)} \quad (3.12)$$

and that

$$\alpha_{n} A_{nn} = j I_{nn}^{(0,j-1)}. \quad (3.13)$$

On integrating by parts, it is not difficult to prove that $P_{nn}^{(1,j)}$ satisfies also the equation

$$\alpha_{n} (j + 1) P_{nn}^{(1,j)} = \alpha_{n} - j \alpha_{n} I_{nn}^{(0,j)} + 2 j A_{nn}. \quad (3.14)$$

Therefore, substituting (3.13) into (3.14) and eliminating $P_{nn}^{(1,j)}$ from (3.12) and (3.14), it follows that

$$\alpha_{n}^{2} (2j + 1) I_{nn}^{(0,j)} = \alpha_{n}^{2} - 2 j I_{nn}^{(0,j-1)}, \quad j = 1, 2, \ldots, M. \quad (3.15)$$

Initially, we have from (2.16)

$$I_{nn}^{(0,0)} = 1 \quad (3.16)$$

for all $n$. 
In a similar fashion, for \( l = 1 \), the integral

\[
I_{nm}^{(1,j)} = c_{nm}^{(1)} \int_0^1 \xi^{2j+1} J_1(\beta_n \xi) J_1(\beta_m \xi) \, d\xi;
\]

\[
n, m = 1, 2, \ldots \quad (3.17)
\]

may be evaluated recursively, where the \( \beta_n \)'s are now the positive zeros of \( J_1(x) \). If \( m \neq n \), we find that

\[
I_{nm}^{(1,j)} = \frac{4j}{(\beta_n^2 - \beta_m^2)^2} \{ 2\beta_n^2 \beta_m^2 [1 - jQ_{nm}^{(0,j-1)}] \\
- (j - 1)(\beta_n^2 + \beta_m^2) I_{nm}^{(1,j-1)} \},
\]

\[
I_{nm}^{(1,0)} = 0 \quad (3.18)
\]

and

\[
Q_{nm}^{(0,j)} = \frac{4j}{(\beta_n^2 - \beta_m^2)^2} \{ (\beta_n^2 + \beta_m^2) [1 - jQ_{nm}^{(0,j-1)}] \\
- 2(j - 1)\beta_n^2 \beta_m^2 I_{nm}^{(1,j-1)} \},
\]

\[
Q_{nm}^{(0,0)} = 0, \quad (3.19)
\]

where

\[
Q_{nm}^{(0,j)} = c_{nm}^{(1)} \int_0^1 \xi^{2j+1} J_0(\beta_n \xi) J_0(\beta_m \xi) \, d\xi,
\]

\[
Q_{nm}^{(0,0)} = Q_{nm}^{(0,j)}. \quad (3.20)
\]

If \( m = n \), then

\[
(2j + 1)\beta_n^2 I_{nm}^{(1,j)} = \beta_n^2 - 2j(j - 1) I_{nm}^{(1,j-1)},
\]

\[
I_{nn}^{(1,0)} = 1. \quad (3.21)
\]

In general, the problem of calculating integrals in (2.14) for any \( l \) requires a treatment of this kind.

4. Applications and Concluding Remarks

The present technique is applied to some specific potentials in (1.6) to show its computational performance. To this end, the infinite system (2.14) has been truncated to a homogeneous system of a finite number of equations, \( N \) say. The truncated eigenvalues are then determined as the roots of the so-called characteristic equation.

We examine numerically the generalized anharmonic oscillators

\[
V(r) = r^2 + \nu_{2k} r^{2k}, \quad k = 2, 3, 4, \quad (4.1)
\]

and the double-well potential of the form

\[
V(r) = -r^2 + d_4 r^4
\]

for a wide range of the coupling constants. As \( \nu_{2k} \to \infty \) in (4.1), we have the infinite-field limit Hamiltonian described by the equation

\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{l^2}{r^2} + r^{2k} \right) \Psi(r) = \lambda \Psi(r),
\]

\[
\lambda = \nu_{2k}^{-1/(k+1)} E. \quad (4.4)
\]

This asymptotic relation shows that the total energy \( E \) of the system (1.3) grows like \( \nu_{2k}^{1/(k+1)} \) for large values of \( \nu_{2k} \). For this reason, we also consider the potentials

\[
V(r) = r^{2k}, \quad k = 1, 2, 3, 4, \text{ and } 10, \quad (4.5)
\]

to cover the limiting case of the anharmonicity constant, as \( \nu_{2k} \to \infty \). The potential in (4.5) with \( k = 1 \), or (4.1) with \( \nu_{2k} = 0 \), corresponds to the harmonic oscillator whose exact eigenstates are given by (1.7) for the unbounded interval of \( r, \quad r \in [0, \infty) \). Therefore, this problem provides a very good check on the accuracy of the present approach, which assumes a finite interval, \( r \in [0, L] \).

As a specimen test, Table I demonstrates the rate of convergence of the method as functions of

<table>
<thead>
<tr>
<th>Table I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convergence rate of the method as functions of ( L ) and ( N ) for the ground-state energy ( E_{0,0} ) of the harmonic oscillator, ( V(r) = r^2 ).</td>
</tr>
<tr>
<td>( L )</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>12</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>16</td>
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<td>18</td>
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<td>22</td>
</tr>
<tr>
<td>24</td>
</tr>
<tr>
<td>11</td>
</tr>
<tr>
<td>28</td>
</tr>
<tr>
<td>30</td>
</tr>
</tbody>
</table>

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the boundary parameter \( L \) and the truncation size \( N \) in calculating the exact ground-state energy, \( E_{0,0} = 2 \), of the harmonic oscillator. It is seen that if we restrict ourselves to about 10 digit accuracy, it is sufficient to set \( L = 5 \). Increasing \( L \) from 5 to 7 and to 9, respectively, 20 and 30 significant figures are obtained.

In our numerical tables, we report eigenvalues to 30 digits. Table I shows clearly that this prescribed accuracy is first achieved at \( L = 9 \), and if we take an \( L \) value beyond \( L = 9 \), for instance, \( L = 11 \), we certainly get the same result again at the cost of using a higher truncation size.

In general, we continue changing \( L \) until the eigenvalues of the required accuracy, i.e., 30 digits, are obtained. The definition of \( L_{cr} \) implies the statement that

\[
|E^{(e)} - E(L_{cr})| < \epsilon,
\]

where \( \epsilon = 10^{-30} \) and \( E(L) \) and \( E^{(e)} \) are the eigenvalues of the bounded and the corresponding unbounded problem, respectively. Also, the influence of the finite boundary on the spectrum, numerically speaking, is greater than \( \epsilon \) when \( L < L_{cr} \). Of course, \( L_{cr} \) is not a unique value which has to be estimated very precisely. Our computational search shows that in the near vicinity of the reported \( L_{cr} \) values the rate of convergence is more or less the same in finding the low-lying state energies. Furthermore, the algorithm employs a minimum number of the basis functions in this neighborhood of \( L_{cr} \). Actually, if \( L \gg L_{cr} \), then a considerable slowing down of convergence may occur.

The existence of an optimum value of \( L \), which should not be very small or very large, may also be deduced from the fact that the kinetic energy
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... term in the variational matrix (2.13) grows unboundedly as \( L \to 0 \), whereas the potential energy term becomes infinite as \( L \to \infty \). On the other hand, the full spectrum of an eigenvalue problem cannot be calculated by estimating a fixed critical distance. Naturally, the larger \( L_{cr} \) values are needed for the higher excited states. As a result of these remarks, we may conclude that \( L_{cr} \) depends on the required accuracy \( \epsilon \), the state number \( n \), and the potential function \( V(r) \) in question.

In Tables II and III, we report the first few state energies of the potentials in (4.5) for \( l = 0 \) and \( l = 1 \), respectively. It is clear that the number of basis functions \( N \) we used increases from 24 to 84 as the potentials vary from harmonic oscillator to the potential \( V(r) = r^3 \). Owing to the contraction of the potentials as \( k \) increases, however, \( L_{cr} \) decreases.

The eigenvalues of the quartic, sextic and octic anharmonic oscillators are tabulated in Tables IV–VI for a very wide range of the coupling constants. It is worth mentioning that there is no accuracy loss in any regime of the eigenvalues. In these tables, the eigenvalues for \( 2k > 1 \) are replaced by \( \frac{1}{2k} \lambda_{n,1} \) to show how rapidly they converge to the \( v_{2k} \to \infty \) limit energies given in Table II. The rate of convergence in each case is consistent with that of the potential in Table II having the same asymptotic behavior. The numerical data are presented only for \( l = 0 \) in order not to overfill the content of the article with tabular material anymore. Further results are available from the authors.

Finally, the method is applied to an eigenvalue problem of a different nature, where the potential has two minima in Cartesian coordinates. We see from Table VII that the method has the capability of yielding accurate results in this case as well. In general, the recorded eigenvalues to 30 digits in our tables are in good agreement with the previously published results to the accuracy quoted [5] (and the references therein).

### Table III

<table>
<thead>
<tr>
<th>( k )</th>
<th>( n )</th>
<th>( \lambda_{n,1} )</th>
<th>( L_{cr} )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>4.000 000 000 000 000 000 000 000 000 000 000 000</td>
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<td>26</td>
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<td>28</td>
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</tr>
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<td>10.5</td>
<td>34</td>
</tr>
</tbody>
</table>

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### TABLE IV
Critical values $L_{cr}$ and the energy eigenvalues $E_{n,0}$ of the quartic oscillator, $V(r) = r^2 + v_4 r^4$, as a function of $v_4$.

<table>
<thead>
<tr>
<th>$v_4$</th>
<th>$n$</th>
<th>$E_{n,0}$</th>
<th>$L_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-3}$</td>
<td>0</td>
<td>2.001 995 522 094 708 533 684 913 20578</td>
<td>10.5</td>
</tr>
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<td></td>
<td>1</td>
<td>6.013 936 098 189 653 073 559 642 74371</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>10.037 726 447 540 990 997 168 029 2097</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>14.073 268 901 848 599 025 546 920 3647</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>18.120 467 854 849 535 519 253 608 5835</td>
<td></td>
</tr>
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<td>$1$</td>
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<td>10.882 435 576 819 807 243 980 305 2898</td>
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<td>20.661 082 690 597 886 008 613 860 6236</td>
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<td>31.725 128 191 323 846 000 694 493 6603</td>
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<td>43.816 823 442 711 769 292 920 245 9907</td>
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<td>2.351 338 918 985 396 323 609 83486</td>
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<td>18.754 903 714 202 645 896 830 091 2136</td>
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<td>29.326 237 058 380 066 510 244 951 0992</td>
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<td>9.529 921 064 696 036 065 423 201 4970</td>
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<td></td>
<td>2</td>
<td>18.735 392 632 619 681 244 508 244 2998</td>
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<td>29.301 795 163 271 981 703 005 355 0114</td>
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<td>4</td>
<td>40.94 221 043 942 410 630 883 781 62603</td>
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</table>

*aEigenvalues for $v_4 > 1$ are $v_4^{-1/3}E_{n,0}$.

### TABLE V
Critical values $L_{cr}$ and the energy eigenvalues $E_{n,0}$ of the sextic oscillator, $V(r) = r^2 + v_6 r^6$, as a function of $v_6$.

<table>
<thead>
<tr>
<th>$v_6$</th>
<th>$n$</th>
<th>$E_{n,0}$</th>
<th>$L_{cr}$</th>
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<td>$10^{-4}$</td>
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<tr>
<td></td>
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<td>10.032 542 763 227 701 804 824 865 7460</td>
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<td>14.086 013 194 042 651 079 974 348 5427</td>
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<td>18.178 227 767 416 675 616 115 325 6809</td>
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<td>3.121 935 474 246 425 991 126 392 39363</td>
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<td>26.720 687 689 389 101 123 735 627 4174</td>
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<td>61.278 364 418 969 761 486 805 218 6638</td>
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</tbody>
</table>

*aEigenvalues for $v_6 > 1$ are $v_6^{-1/4}E_{n,0}$.
In this article, to the best of our knowledge, a Fourier–Bessel expansion for the wave function is utilized for the first time for solving eigenvalue problems of this kind. The accuracy of the method is quite impressive. The evaluation of the matrix elements given in Section 3 requires only the determination of the zeros of Bessel functions accurately. To this end, making use of Mathematica [9], we calculated the zeros to 40 digits.

It is well known, from the variational principle, that the Rayleigh–Ritz method provides an upper bound for the eigenvalues. Therefore, better ap-

<table>
<thead>
<tr>
<th>TABLE VI</th>
<th>Critical values $L_{cr}$ and the energy eigenvalues $E_{n,0}$ of the octic oscillator, $V(r) = r^2 + v_8 r^8$, as a function of $v_8$.</th>
</tr>
</thead>
<tbody>
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</table>

$^a$Eigenvalues for $v_8 > 1$ are $v_8^{-1/5}E_{n,0}$. 

<table>
<thead>
<tr>
<th>TABLE VII</th>
<th>Critical values $L_{cr}$ and the energy eigenvalues $E_{n,0}$ of the two-well potential, $V(r) = -r^2 + d_4 r^4$, as a function of $d_4$.</th>
</tr>
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<td>$d_4 , ^a$</td>
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</table>
proximations are sure to be achieved for successive values of $N$. On the other hand, it is not surprising that the larger region has smaller eigenvalues for such a boundary value problem. Actually, Nunez [10] showed theoretically that the eigenfunctions of an unbounded system can be approximated by means of the numerical solutions of the Dirichlet problem in the Hilbert space $L^2(\Omega)$ with sufficiently large $\Omega$, where $\Omega$ is a bounded region. This makes our strategy plausible in obtaining the eigenvalues of unbounded oscillators by increasing the boundary parameter $L$. However, our studies on the estimation of error bounds and the generalization of the method to higher-dimensional spaces are in progress and will be reported in the near future.

References