MATH 594 Theory of Special Functions

Homework 3 (Hypergeometric Functions)

Exercise 1

(a) Verify that the EHT \((z - a)y'' + \tau(z)y' + \lambda y = 0\) in case 2 of Section 2.1 leads to a particular Lommel equation introduced in HW 1, when \(\tau' = 0\).

(b) Consider the general form of the Lommel equation

\[ s^2y'' + (1 - 2\alpha)sy' + [(\gamma/\beta s)^2 + \alpha^2 - \nu^2]\gamma^2y = 0 \]

First, show that if \(\xi = \beta s^\gamma\) then the following operational identities are obtained:

\[ s \frac{d}{ds} = \gamma \xi \frac{d}{d\xi} \quad \text{and} \quad s^2 \frac{d^2}{ds^2} = \gamma^2 \xi^2 \frac{d^2}{d\xi^2} + \gamma(\gamma - 1)\xi \frac{d}{d\xi} \]

Then show that the substitution \(y(\xi) = \xi^\kappa u(\xi)\) transforms the Lommel equation into the Bessel equation in Example 1.1

\[ \frac{d^2u}{d\xi^2} + \frac{1}{\xi} \frac{du}{d\xi} + \left(1 - \frac{\nu^2}{\xi^2}\right)u = 0 \]

providing the parameter \(\kappa = \alpha/\gamma\).

(c) Let \(u_\nu(\xi)\) be a solution of the Bessel equation in Part (b), which is called, in general, the Bessel function of order \(\nu\). Then give an expression for a solution of the Lommel equation, in Part (b), and, hence, an expression for a solution of the EHT in Part (a).

(d) Find the general solution of the equation

\[ (x + 1)y'' - y = 2x - x^2, \quad x \in (0, 1) \]

Then find the exact solution of the BVP, where \(y(0) = 1\) and \(y(1) = 2\).

Exercise 2 Show that the Gauss HG equation is reduced to the confluent HG equation as a limiting case of the parameter \(\beta \to \infty\).
Exercise 3 Find the weight function $\rho(z)$ in each case, which is necessary for a consideration of the Gauss HG, confluent HG, Hermite equations in formal self-adjoint (or Sturm-Liouville) forms.

Exercise 4 Find the parameter values for which the Gauss HG, confluent HG and Hermite equations have polynomial solutions.

Exercise 5 Recalling that the solution $u_1(z)$ and $u_2(z)$ of the Gauss HG equation in (2.2.2) are linearly independent, show that

$$u_1(z) = u_3(z)$$

and

$$u_2(z) = u_4(z)$$

for $\text{Re}\gamma > 1$.

Exercise 6 Show that the contours $s = 1 - (1 - z)t$ and $s = z/t$ in (2.3.4)(ii) and (iii) lead to the pairs of linearly independent solutions

$$\begin{cases} u_1(z) = f(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z) \\ u_2(z) = (1 - z)^{\gamma - \alpha - \beta} f(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1; 1 - z) \end{cases} (2.3.10)$$

and

$$\begin{cases} u_1(z) = z^{-\alpha} f(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1; 1/z) \\ u_2(z) = z^{-\beta} f(\beta - \gamma + 1, \beta - \alpha + 1; 1/z) \end{cases} (2.3.11)$$

do the Gauss HG equation, respectively, where the parameter $t$ runs from 0 to 1.

Exercise 7 Show that the Hermite functions $H_\nu(z)$ and $H_\nu(-z)$ in (2.3.19) are linearly independent provided that $\nu$ is NOT an integer.