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MATH 594 Theory of Special Functions

Homework 3 (Hypergeometric Functions)

Exercise 1

(a) Verify that the EHT $(z-a)y'' + \tau(z)y' + \lambda y = 0$ in case 2 of Section 2.1 leads to a particular Lommel equation introduced in HW 1, when $\tau' = 0$.

(b) Consider the general form of the Lommel equation

$$s^2 y'' + (1-2\alpha)sy' + [(\gamma\beta s^\gamma)^2 + \alpha^2 - \nu^2\gamma^2]y = 0$$

First, show that if $\xi = \beta s^\gamma$ then the following operational identities are obtained:

$$s\frac{\mathrm{d}}{\mathrm{d}s} = \gamma\xi\frac{\mathrm{d}}{\mathrm{d}\xi} \qquad \text{and} \qquad s^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} = \gamma^2\xi^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \gamma(\gamma-1)\xi\frac{\mathrm{d}}{\mathrm{d}\xi}$$

Then show that the substitution $y(\xi) = \xi^\kappa u(\xi)$ transforms the Lommel equation into the Bessel equation in Example 1.1

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}u}{\mathrm{d}\xi} + \left(1 - \frac{\nu^2}{\xi^2}\right)u = 0$$

providing the parameter $\kappa = \alpha/\gamma$.

(c) Let $u_\nu(\xi)$ be a solution of the Bessel equation in Part (b), which is called, in general, the Bessel function of order ν. Then give an expression for a solution of the Lommel equation, in Part (b), and, hence, an expression for a solution of the EHT in Part (a).

(d) Find the general solution of the equation

$$(x+1)y'' - y = 2x - x^2, \quad x \in (0,1)$$

Then find the exact solution of the BVP, where $y(0) = 1$ and $y(1) = 2$.

Exercise 2 Show that the Gauss HG equation is reduced to the confluent HG equation as a limiting case of the parameter $\beta \to \infty$.

Exercise 3 Find the weight function $\rho(z)$ in each case, which is necessary for a consideration of the Gauss HG, confluent HG, Hermite equations in formal self-adjoint (or Sturm-Liouville) forms.

Exercise 4 Find the parameter values for which the Gauss HG, confluent HG and Hermite equations have polynomial solutions.

Exercise 5 Recalling that the solution $u_1(z)$ and $u_2(z)$ of the Gauss HG equation in (2.2.2) are linearly independent, show that

$$u_1(z) = u_3(z)$$

and

$$u_2(z) = u_4(z)$$

for $\mathrm{Re}\gamma > 1$.

Exercise 6 Show that the contours $s = 1-(1-z)t$ and $s = z/t$ in (2.3.4)(ii) and (iii) lead to the pairs of linearly independent solutions

$$\begin{cases} u_1(z) = f(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z) \\ u_2(z) = (1-z)^{\gamma-\alpha-\beta} f(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z) \end{cases} \tag{2.3.10}$$

and

$$\begin{cases} u_1(z) = z^{-\alpha} f(\alpha, \alpha-\gamma+1, \alpha-\beta+1; 1/z) \\ u_2(z) = z^{-\beta} f(\beta, \beta-\gamma+1, \beta-\alpha+1; 1/z) \end{cases} \tag{2.3.11}$$

of the Gauss HG equation, respectively, where the parameter t runs from 0 to 1.

Exercise 7 Show that the Hermite functions $\mathrm{H}_\nu(z)$ and $\mathrm{H}_\nu(-z)$ in (2.3.19) are linearly independent provided that ν is NOT an integer.