EXERCISE SET #5

1. Obtain the Euler equation that governs the stationary function $u(x)$ of the functional

$$I[u]=\int_0^1\left[(u')^2+uu'+u^2\right]dx\,.$$

   (a) Assume that the values of $u(x)$ at the boundaries are specified.

   (b) Assume that the values of $u(x)$ at the boundaries (essential b.c.) are not specified.

2. Show that no continuous stationary function $u(x)$ exists for the functional

$$I[u]=\int_0^1\left[u^2+xu+2u^2u'\right]dx$$

   with the boundary conditions $u(0)=1$ and $u(1)=2$.

3. Determine the stationary function $u(x)$ for the dual functional

$$I[u]=u(1)+u(0)u'(0)+\tfrac{1}{2}\int_0^1(u')^2dx\,.$$

4. Determine the general form of the stationary function $u(x)$ for the functional

$$I[u]=\int_0^1\left[16u^2-(u'')^2+x^2\right]dx$$

   where the boundary conditions are such that the values of $u(x)$ and derivatives of $u(x)$ are all specified.

5. Convert the differential equation

$$u''-u=0,\quad u(0)=0,\quad u(1)=0$$

   into its equivalent variational form.

6. Obtain the Euler equation for the functional

$$I[u]=\iint_D\left[a(x,y)\left(\tfrac{\partial u}{\partial x}\right)^2+b(x,y)\left(\tfrac{\partial u}{\partial y}\right)^2-c(x,y)u^2\right]dxdy\,.$$

7. Consider the functional

$$I[u]=\iiint_V\left[\left(\tfrac{\partial u}{\partial x}\right)^2+\left(\tfrac{\partial u}{\partial y}\right)^2+\left(\tfrac{\partial u}{\partial z}\right)^2+2fu\right]dxdydz\,.$$

   where $f=f(x,y,z)$ is a specified function. Show that the stationary function $u(x)$ must satisfy the Poisson equation.

8. Consider the functional

$$I[u]=\tfrac{1}{2}\int_0^1\int_0^{2\pi}\left[\left(\tfrac{\partial u}{\partial r}\right)^2+\tfrac{1}{r^2}\left(\tfrac{\partial u}{\partial \theta}\right)^2\right]r\,d\theta\,dr$$

   with the boundary condition $u(1,\theta)=u_0(\theta)$, where $u_0(\theta)$ is specified. Show that the stationary function $u(x)$ must satisfy the Laplace equation in polar coordinates.

9. The total kinetic and potential energy of a rotating elastic bar is given by the functional

$$I[u]=\int_{t_1}^{t_2}\int_0^1\left[\left(\tfrac{\partial u}{\partial t}\right)^2+\Omega^2(r+u)^2-K^2\left(\tfrac{\partial u}{\partial r}\right)^2\right]dr\,dt$$

where $u(r,t)$ is the radial displacement, Ω is the constant angular velocity, and K is a constant. The radial location of the end of the bar at $r=0$ is fixed, in which case $u(0,t)=0$. Obtain Euler equation, i.e. the equation of motion, along with the boundary conditions at $r=0$ and $r=1$.

10. Consider the transverse deflection of a string of length L with mass per unit length ρ subject to a transverse load $f(x,t)$ and constant tension force P. The governing differential equation is

$$\rho\frac{\partial^2 u}{\partial t^2}-P\frac{\partial^2 u}{\partial x^2}-f(x,t)=0,$$

where $u(x,t)$ is the transverse deflection of the string. Show that the corresponding variational form is

$$I[u]=\int_{t_1}^{t_2}\int_0^L\left[\tfrac{1}{2}\rho\left(\tfrac{\partial u}{\partial t}\right)^2-\tfrac{1}{2}P\left(\tfrac{\partial u}{\partial x}\right)^2+fu\right]dx\,dt\,.$$

11. Consider the functional

$$I[u,v]=\tfrac{1}{2}\int_0^1\left[\left(u'\right)^2+\left(v'\right)^2+2u'v'\right]dx$$

with the boundary conditions $u(0)=0$, $v(1)=1$, and natural boundary conditions. Obtain the Euler equations and all necessary boundary conditions for $u(x)$ and $v(x)$.

12. In the context of 2D grid generation, there arises the weighted-length functional

$$I[u,v]=\tfrac{1}{2}\iint_D\left\{\tfrac{1}{\phi}\left[\left(\tfrac{\partial u}{\partial x}\right)^2+\left(\tfrac{\partial v}{\partial x}\right)^2\right]+\tfrac{1}{\psi}\left[\left(\tfrac{\partial u}{\partial y}\right)^2+\left(\tfrac{\partial v}{\partial y}\right)^2\right]\right\}dxdy,$$

where the two weight functions $\phi(x,y)>0$ and $\psi(x,y)>0$ are given. Show that the resulting Euler equations are

$$\frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\psi}\frac{\partial u}{\partial y}\right)=0,\quad \frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\psi}\frac{\partial v}{\partial y}\right)=0\,.$$

13. Determine the Euler equation satisfied by the stationary function $u(x)$ for the functional

$$I[u]=\int_0^1\left[\left(\sqrt{x}u'\right)^2-\left(xu\right)^2\right]dx\,,$$

subject to the boundary conditions $u(0)=0$, $u(1)=0$, and the constraint

$$\int_0^1\left(xu\right)^2dx=1\,.$$

14. Consider the functional

$$I[u]=\int_{x_0}^{x_1}\left[s(x)\left(u''\right)^2-p(x)\left(u'\right)^2+q(x)u^2\right]dx\,,$$

subject to the constraint

$$\int_{x_0}^{x_1} r(x)u^2 dx = 1 .$$

If $u(x)$ and its derivatives are specified at the boundaries, show that the corresponding Euler equation is the Sturm-Liouville equation

$$\frac{d^2}{dx^2}\left(s(x)\frac{d^2u}{dx^2}\right)+\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+\left[q(x)+\lambda r(x)\right]u(x)=0 ,$$

where λ is a constant.

15. Determine the stationary functions $u(x)$ and $v(x)$ for the dual functional

$$I[u,v]=\tfrac{1}{2}\left[u(\pi)\right]^2+\int_0^{\pi}\left\{\tfrac{1}{2}\left[(u')^2+(v')^2\right]+1\right\}dx$$

subject to the boundary conditions $u(0)=0$, $v(0)=0$, $u(\pi)+v(\pi)=1$, with the constraint $u'-v=0$. Because this is a dual functional, impose the boundary conditions as constraints.

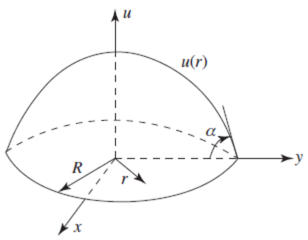
16. Fermat's principle of optics states that the path of light between two points is the one that requires the minimum travel time. This requires to minimize the travel time T[u(x)] of light along all the possible paths $u(x)$ that the light could take through a medium between the points x_0 and x_1 at times t_0 and t_1, respectively. Thus, the functional

$$T[u(x)]=\int_{t_0}^{t_1} dt=\int_{x_0}^{x_1}\frac{ds}{V(x,u)}=\int_{x_0}^{x_1}\frac{n(x,u)ds}{c}$$

where $V(x,u)$ is the speed of light in the medium, c is the speed of light in a vacuum, $n(x,u)$ is the index of refraction in the medium, and ds is a differential element along the path $u(x)$, i.e. $ds=\sqrt{1+(u')^2}dx$.

Determine the general path of light in a medium in which the index of refraction only depends upon the vertical coordinate such that $n(u)=\sqrt{u}$. The starting and ending points of the light path are (x_0,u_0) and (x_1,u_1), respectively.

17. The total energy of a liquid drop when placed on a smooth horizontal surface



is given by

$$E[u(r)]=E_p+E_s=\pi\int_{r=R}^{r=0}\left\{\rho g r^2 u(r)u'(r)+2\sigma r\sqrt{1+[u'(r)]^2}\right\}dr .$$

Here, E_p is the potential energy, E_s is the surface energy, ρ is the density of the liquid, g is the acceleration due to gravity, σ is the surface energy per unit area and u(r) is the vertical distance to the interface from the surface. The axisymmetric drop shape u(r) results in minimization of the total energy E[u(r)] subject to the constraint that the total volume of the liquid drop is fixed and given by

$$\bar{V} = \pi\int_{r=R}^{r=0} r^2 u'(r)dr .$$

Show that the Euler equation to be solved in order to minimize the total energy of the liquid drop is

$$\frac{u''}{\left[1+(u')^2\right]^{3/2}} + \frac{u'}{r\left[1+(u')^2\right]^{1/2}} + \frac{\rho g}{\sigma}u = -\frac{\lambda}{\sigma},$$

subject to the boundary conditions $u'(0)=0$, $u(R)=0$, $u'(R)=-\tan\alpha$, where α is the given contact angle. The constraint on the bubble volume and the extra boundary condition allow us to find the Lagrange multiplier λ and the bubble radius R, respectively.

18. Consider the variational problem for which the left endpoint is a fixed point A while the right hand endpoint moves freely on a given boundary curve C parametrized by t with its position vector $\mathbf{R}(t) = (x_r(t), u_r(t))$.

In order to construct a variational formulation, let us introduce a new notation as follows: Consider the set of functions $u(x;\varepsilon)$ defined for $[a,b]$ by

$$u(x;\varepsilon) = u_0(x) + \varepsilon\eta(x), \quad |\varepsilon| < \varepsilon_0$$

where ε_0 is a positive, sufficiently small number and $\eta(x)$ is a smooth function on $[a,b]$. For a given function η, the value of the functional $I[u(x;\varepsilon)]$ depends only on the variable ε, thus the variational problem is stated as follows:

$$\Phi(\varepsilon) = I[u(x;\varepsilon)] = \int_a^b F(x, u(x;\varepsilon), u_x(x;\varepsilon))\,dx ,$$

For the stationary function $u_0(x) = u(x;0)$, the first variation of the functional

$$\Phi'(0) = \int_a^b \left[F_u^0(x)\eta(x) + F_{u'}^0(x)\eta'(x)\right]dx$$

must vanish. Here

$$F_u^0(x) = \frac{\partial F}{\partial u}(x, u_0(x), u_0'(x)) \quad \text{and} \quad F_{u'}^0(x) = \frac{\partial F}{\partial u'}(x, u_0(x), u_0'(x)) .$$

This leads to

$$\Phi'(0) = \int_a^b \left[F_u^0(x) - \tfrac{d}{dx}F_{u'}^0(x)\right]\eta(x)\,dx + \left[F_{u'}^0(x)\eta(x)\right]_a^b = 0 .$$

For essential b.c. $\eta(a) = \eta(b) = 0$, the Euler equation

$$F_u^0(x) - \tfrac{d}{dx}F_{u'}^0(x) = 0$$

results.

Consider now the variational problem with the variable right hand endpoint. The right hand endpoint of the curve $u(x;\varepsilon)$ lies on the boundary curve C so that the domain of $u(\cdot;\varepsilon)$ is $[a, x_r(t(\varepsilon))]$ where movement along the curve C as ε varies is indicated by $t(\varepsilon)$. Thus, the variational problem with the variable right hand endpoint is stated as

$$\Phi(\varepsilon) = \int_a^{x_r(t(\varepsilon))} F(x, u(x;\varepsilon), u_x(x;\varepsilon))\, dx,$$

(a) Show that the first variation leads to

$$\Phi'(0) = F(b, u_0(b), u_0'(b)) \frac{dx_r}{dt}\frac{dt}{d\varepsilon}(0) + \int_a^b \left[F_u^0(x)\frac{\partial u}{\partial \varepsilon}(x;0) + F_{u'}^0(x)\frac{\partial^2 u}{\partial\varepsilon\partial x}(x;0) \right] dx.$$

Its vanishing results in the Euler equation

$$F_u^0(x) - \tfrac{d}{dx} F_{u'}^0(x) = 0$$

and the transversality condition

$$F_{u'}^0(b) u_r'(t_0) + \left[F(b, u_0(b), u_0'(b)) - F_{u'}^0(b) u_0'(b) \right] x_r'(t_0) = 0$$

where $t_0 = t(\varepsilon = 0)$ is the parameter of the right hand endpoint of u_0.

(b) Show that for the variational problem of the shortest curve joining a point A and a curve C in the plane, then curve C cuts the stationary function u_0 transversally precisely when C cuts the graph of u_0, orthogonally.

Recall that the integrand F is given by $F(x,u,u') = \sqrt{1+(u')^2}$.

(c) Among all piecewise smooth curves $u(x;\varepsilon)$ which join the point $A(a,0)$ with a point P on the semi-circle C having radius R, i.e.

$$C = \left\{ (x,u) \mid x_r(t) = R\cos t, \quad u_r(t) = R\sin t, \quad -\pi/2 < t < \pi/2 \right\},$$

find the curve having the shortest length.

Recall that the extremals of the variational problem are straight line segments.

19. Consider the functional

$$I[u] = \int_0^1 \left[(xu')^2 + 2u^2 \right] dx$$

with the boundary conditions $u(0) = 0$, $u(1) = 2$.

(a) Use the Raylegh-Ritz method, with the trial function $\tilde{u}(x) = 2x + x(1-x)(c_1 + c_2 x)$ to determine an approximation to the stationary function $u(x)$. Note that $\phi_0(x) = 2x$ is chosen to satisfy the boundary conditions.

(b) Compare the approximate solution from part (a) with the exact solution.

20. Consider the differential equation

$$\frac{d}{dx}\left(x\frac{du}{dx}\right)+u=x$$

along with the boundary conditions $u(0)=0$, $u(1)=1$.

(a) Convert the differential equation into its equivalent variational form.

(b) Use the Rayleigh-Ritz method to find an approximate solution with the trial function $\tilde{u}(x)=x+c_1x(1-x)$. Note that $\phi_0(x)=x$ is chosen to satisfy the boundary conditions.

21. Consider the differential equation

$$\frac{d}{dx}\left((1+x)\frac{du}{dx}\right)+\lambda u=0$$

along with the boundary conditions $u(0)=0$, $u(1)=0$. Obtain an approximation to the smallest eigenvalue λ using the trial function $\tilde{u}(x)=c_1x(1-x)$.