EXERCISE SET #3

Theorem: Let D be a closed region between, and including circles with their centers at $z = a$. If $f(z)$ is analytic in D, then it admits the Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

in D, with the c_n given uniquely by

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}$$

where C is any piecewise smooth simple closed anticlockwise path in D.

Definition: Let $f(z)$ be singular (not analytic) at $z = a$. If it is analytic in an annulus $0 < |z-a| < \rho$ for some $\rho > 0$, then $z = a$ is said to be an isolated singular point of f, otherwise it is a nonisolated singular point.

Definition: If $f(z)$ has an isolated singularity at $z = a$, then f admits a Laurent series representation about $z = a$ in the annulus $0 < |z-a| < \rho$. If the Laurent expansion terminates on the left, i.e. $c_{n<-N} = 0$, then the singularity of f at $z = a$ is classified as an Nth order pole (simple pole for $N = 1$). If not, the singularity is classified as an essential singularity.

Residue Theorem: Let C be a piecewise smooth simple closed curve oriented anticlockwise and let $f(z)$ be analytic inside and on C except at finitely many isolated points $z_1,...,z_k$ within C. If $c_{-1}^{(j)}$ denotes the residue of f at z_j, then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} c_{-1}^{(j)}$$

where the reside at an Nth order pole is calculated by

$$c_{-1} = \frac{1}{(N-1)!}\lim_{z\to a}\left\{\frac{d^{N-1}}{dz^{N-1}}\left[(z-a)^N f(z)\right]\right\}.$$

1. Obtain the Taylor series of the given function about $z = z_0$ and give its radius of convergence R.

   (a) $\sin z;\quad z_0 = 0$, (b) $\sin z;\quad z_0 = 2-i$, (c) $\cos 2z;\quad z_0 = 3i$, (d) $e^{z^6};\quad z_0 = 0$,

   (e) $\frac{1}{i+z};\quad z_0 = 0$, (f) $\frac{z^3}{2-iz};\quad z_0 = 0$, (g) $\sin z^8;\quad z_0 = 0$, (h) $z^3;\quad z_0 = -2i$,

   (i) $\frac{1}{1+2z^{35}};\quad z_0 = 0$, (j) $z^2 - iz;\quad z_0 = 2i$, (k) $\frac{1}{(3-z)^2};\quad z_0 = 0$, (l) $\frac{1}{(2z+1)^3};\quad z_0 = 0$,

   (m) $\frac{1}{(2z+1)^3};\quad z_0 = 2$, (n) $\frac{1}{z^2-z-6};\quad z_0 = -1$, (o) $\tan z = \frac{\sin z}{\cos z};\quad z_0 = 0$, (p) $\sec z = \frac{1}{\cos z};\quad z_0 = 0$,

   (q) $\operatorname{cosec} z = \frac{1}{\sin z};\quad z_0 = 0$, (r) $\frac{1+z}{1+2z+3z^2};\quad z_0 = 0$, (s) $\frac{3-z}{2+3z^2+z^4};\quad z_0 = 0$, (t) $\frac{e^z}{\sin 2z};\quad z_0 = 0$,

   (u) $\frac{1}{2-\sin z};\quad z_0 = 0$, (v) $\frac{1}{3+\cos z};\quad z_0 = 0$.

   Hint: The binomial series expansion $\frac{1}{(1-z)^m} = \sum_{n=0}^{\infty}\frac{(m+n-1)!}{(m-1)!n!}z^n$, $|z| < 1$ for any positive integer m and the method of undetermined coefficients for determining the expansion for $f(z) = \frac{g(z)}{h(z)} = \sum_{n=0}^{\infty} a_n z^n$ by using the product $\sum_{n=0}^{\infty} b_n z^n = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} c_n z^n\right)$ for given $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and $h(z) = \sum_{n=0}^{\infty} c_n z^n$, may be useful in this exercise.

Ans.: (a) $z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \ldots$; $R = \infty$, (b) $\sin a + \cos a(z-a) - \frac{\sin a}{2!}(z-a)^2 - \frac{\cos a}{3!}(z-a)^3 + \frac{\sin a}{4!}(z-a)^4 + \ldots$; $R = \infty$,
(c) $\cosh 6 - 2i(\sinh 6)(z-3i) - 2(\cosh 6)(z-3i)^2 + \frac{4}{3}i(\sinh 6)(z-3i)^3 + \ldots$; $R = \infty$, (d) $1 + z^6 + \frac{1}{2!}z^{12} + \frac{1}{3!}z^{18} + \ldots$; $R = \infty$,
(e) $-i + z + iz^2 - z^3 - \ldots$ $R = 1$, (f) $\frac{1}{2}z^3 + \frac{i}{4}z^4 - \frac{1}{8}z^5 - \frac{i}{16}z^6 - \ldots$ $R = 2$, (g) $z^8 - \frac{1}{3!}z^{24} + \frac{1}{5!}z^{40} - \ldots$; $R = \infty$,
(h) $(-2i)^3 + 3(-2i)^2(z+2i) - \frac{6}{2!}(-2i)(z+2i)^2 + \frac{6}{3!}(z+2i)^3$; $R = \infty$, (i) $1 - 2z^{35} + 4z^{70} - 8z^{105} + \ldots$ $R = 1/2^{1/35}$,
(j) $-2 + 3i(z-2i) + (z-2i)^2$; $R = \infty$, (k) $\frac{1}{(3-i)^2}\sum_{n=0}^{\infty}(n+1)(\frac{z-i}{3-i})^n$; $|z-i| < |3-i|$, (l) $\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)2^n z^n$; $|z| < 1/2$,
(m) $\frac{1}{250}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)(\frac{2}{5})^n(z-2)^n$; $|z-2| < 5/2$, (n) $\frac{1}{20}\sum_{n=0}^{\infty}\frac{(-4)^{n+1}-1}{4^n}(z+1)^n$; $|z+1| < 1$,
(o) $z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \ldots$ $|z| < \pi/2$, (p) $1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \ldots$ $|z| < \pi/2$, (q) Not analytic at 0, (r) $1 - z - z^2 + 5z^3 - 7z^4 + \ldots$ $|z| < 1/\sqrt{3}$,
(s) $\frac{3}{2} - \frac{1}{2}z - \frac{9}{4}z^2 + \frac{3}{4}z^3 + \frac{21}{8}z^4 - \ldots$ $|z| < 1$, (t) Not analytic at 0, (v) $\frac{1}{4} + \frac{1}{32}z^2 + \frac{1}{768}z^4 + \ldots$ $R = \sqrt{\pi^2 + (\ln(3+2\sqrt{2}))^2}$.

2. Determine all possible Taylor and Laurent expansions about the given point $z = z_0$ and state their regions of validity.

(a) $\frac{1}{z}$; $z_0 = i$, (b) $\frac{1}{z^2+1}$; $z_0 = 0$, (c) $\frac{z^2+3}{z}$; $z_0 = 0$, (d) $\frac{1}{e^z-1}$; $z_0 = 0$, (e) $\frac{1}{z(z^3+2)}$; $z_0 = 0$,
(f) $\frac{1}{z} + \frac{z}{z+i}$; $z_0 = 0$, (g) $\frac{1}{z^3}$; $z_0 = 2$, (h) $\frac{1}{z^2}$; $z_0 = -i$, (i) $\frac{1}{\cos z}$; $z_0 = \pi/2$,
(j) $\tan z$; $z_0 = -\pi/2$, (k) $\sin\frac{1}{z}$; $z_0 = 0$, (l) $\frac{1}{z}$; $z_0 = -2$, (m) e^{-1/z^3}; $z_0 = 0$,
(n) $\frac{z^2+5}{z^2+4}$; $z_0 = -1$, (o) $\frac{1+z}{2+z}$; $z_0 = -i$, (p) $\frac{\sin z}{z^4}$; $z_0 = 0$, (q) $\frac{\cos 2z}{(z+i)^2}$; $z_0 = -i$,
(r) e^{-z^2}; $z_0 = 0$, (s) $e^{-1/z}$; $z_0 = 0$, (t) $\frac{1}{z(z^2+1)}$; $z_0 = 1$, (u) $\frac{1}{z^2+iz+2}$; $z_0 = i$,
(v) $\frac{1}{z^2}$; $z_0 = 1+i$

Ans.: (a) $\frac{1}{z-i} - \frac{i}{(z-i)^2} - \frac{1}{(z-i)^3} + \frac{i}{(z-i)^4} - \ldots$, (b) $\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2(n+1)}}$, (c) $\frac{3}{z} + z$, (d) $\frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + \ldots$, (e) $\frac{1}{2z} - \frac{1}{4}z^2 + \frac{1}{8}z^5 - \frac{1}{16}z^8 + \ldots$,
(f) $1 + \frac{1-i}{z} - \frac{1}{z^2} + \frac{i}{z^3} - \ldots$, (g) $\frac{1}{(z-2)^3} - \frac{6}{(z-2)^4} + \frac{24}{(z-2)^5} - \frac{80}{(z-2)^6} + \ldots$, (h) $\sum_{n=0}^{\infty}\frac{(n+1)i^n}{(z+i)^{n+2}}$, (i) $-\frac{1}{z-\frac{\pi}{2}} - \frac{1}{6}\frac{1}{(z-\frac{\pi}{2})^3} - \frac{7}{360}\frac{1}{(z-\frac{\pi}{2})^5} - \ldots$,
(k) $\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \ldots$ $0 < |z| < \infty$, (l) $\frac{1}{2} - \frac{(z+2)}{4} - \frac{(z+2)^2}{8} - \ldots$ $0 \le |z+2| < 2$; $\frac{1}{(z+2)} + \frac{1}{(z+2)^2} + \ldots$ $2 < |z+2| < \infty$, (m) $1 - \frac{1}{z^3} + \frac{1}{2!}\frac{1}{z^6} - \frac{1}{3!}\frac{1}{z^9} + \ldots$
$0 < |z| < \infty$, (n) $\frac{6}{5} + \frac{2}{25}(z+1) - \frac{1}{125}(z+1)^2 - \ldots$ $0 < |z+1| < \sqrt{5}$; $1 - \frac{7(1+i)}{4}\frac{1}{(z+1)} + \ldots$ $\sqrt{5} < |z+1| < \infty$,
(o) $\frac{3-i}{5} - \frac{3+4i}{25}(z+i) + \frac{2+11i}{125}(z+i)^2 - \ldots$ $0 < |z+i| < \sqrt{5}$; $1 - \frac{1}{(z+i)} + \frac{2-i}{(z+i)^2} - \ldots$ $\sqrt{5} < |z+i| < \infty$, (p) $\frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{1}{120}z - \ldots$ $0 < |z| < \infty$,
(q) $\frac{\cosh 2}{(z+i)^2} + \frac{2i\sinh 2}{z+i} - 2\cosh 2 + \frac{4i\sinh 2}{3}(z+i) - \ldots$ $0 < |z+i| < \infty$, (r) $1 - z^2 + \frac{1}{2!}z^4 - \frac{1}{3!}z^6 + \ldots$ $0 < |z| < \infty$, (s) $1 - \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} - \frac{1}{3!}\frac{1}{z^3} + \ldots$ $0 < |z| < \infty$
(t) Let $\frac{1}{z(z^2+1)} = \frac{1}{z} - \frac{1}{2}\frac{1}{z+i} - \frac{1}{2}\frac{1}{z-i}$ to get $\frac{1}{2} - (z-1) + \frac{5}{4}(z-1)^2 - \ldots$ $0 < |z-1| < 1$; $\ldots - \frac{1}{(z-1)^2} + \frac{1}{(z-1)} - \frac{1}{2} + \frac{1}{4}(z-1)^2 - \frac{1}{4}(z-1)^3 + \ldots$ $1 < |z-1| < \sqrt{2}$

3. The generating function for the Bessel functions $J_n(t)$ is $\exp(\frac{t}{2}(z - \frac{1}{z})) = \sum_{n=-\infty}^{\infty} J_n(t) z^n$ where t is an independent real variable. Derive the integral representation of $J_n(t)$

$$J_n(t) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - t\sin\theta)\,d\theta.$$

Hint: Consider Laurent series expansion for the generating function $f(z;t) = \exp(\frac{t}{2}(z - \frac{1}{z})) = \sum_{n=-\infty}^{\infty} a_n(t) z^n$ whose expansion coefficients are given $a_n(t) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta;t)}{\zeta^{n+1}}d\zeta$ for C taken to be the unit circle.

4. Determine all singular points in the extended complex plane, i.e. including ∞, of the following functions. If isolated, classify them further as Nth order poles or essential singularities.

(a) $\frac{e^z-1}{z^3}$, (b) $\frac{z^2}{e^z-1}$, (c) $\frac{1}{z^3-1}$, (d) $\frac{1}{1+1/(1+z)}$, (e) $\frac{1}{\sinh z}$, (f) $\frac{1}{\cosh z}$, (g) $\frac{z}{\sin^3 z}$, (h) $\sin\frac{1}{z}$, (i) $\cos\frac{1}{z}$,

(j) $\sinh\frac{1}{z}$, (k) $\cosh\frac{1}{z}$, (l) $\sin\frac{1}{z^3}$, (m) e^{1/z^3}, (n) $4-\frac{z}{(z-1)^2}$, (o) $\frac{1}{e^z}$, (p) $\tan z^2$, (q) $\tan\frac{1}{z^2}$,

(r) $\frac{1}{e^{z^2}}$, (s) $\frac{1}{\sin(z-2)}$, (t) $\frac{1}{\sin\frac{1}{z}}$, (u) $\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\dots$, (v) $\frac{1}{2z}-\frac{1}{(2z)^2}+\frac{1}{(2z)^3}-\dots$, (w) $\frac{2}{z^4}+3z^4$,

(x) $\frac{1}{2!z^2}+\frac{1}{3!z^3}+\frac{1}{4!z^4}+\dots$, (y) $\frac{1}{z^5}\left(1-\left(\frac{2}{z^3}\right)+\left(\frac{2}{z^3}\right)^2-\left(\frac{2}{z^3}\right)^3+\dots\right)$

Ans.: (a) 2nd order pole at 0; essential singularity at ∞, (b) 1st order poles at $2n\pi i$, ($n=\pm1,\pm2,\dots$); essential singularity at ∞, (c) 1st order poles at $1, \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$, (d) 1st order pole at -2, (e) 1st order poles at $n\pi i$ ($n=0,\pm1,\pm2,\dots$), (f) 1st order poles at $n\pi i/2$ ($n=\pm1,\pm2,\dots$), (g) 2nd order pole at 0, 3rd order poles at $n\pi$ ($n=\pm1,\pm2,\dots$), (h-m) essential singularity at 0, (n) 2nd order pole at 1, (o) none, (p) 1st order poles at $z^2=n\pi/2$ ($n=\pm1,\pm3,\dots$),(q) 1st order poles at $z^2=2/n\pi$ ($n=\pm1,\pm3,\dots$), (r) none, (s) 1st order poles at $2+n\pi$ ($n=0,\pm1,\pm2,\dots$), (t) 1st order poles at $1/n\pi$ ($n=\pm1,\pm2,\dots$), (u) 1st order pole at 0, (v) analytic at 0, (w) 4th order pole at 0, (x) essential singularity at 0, (y) 2nd order pole at 0.

5. Evaluate the given integral by means of the residue theorem.

(a) $\oint_{C_1}\frac{dz}{\sin 2z}$, (b) $\oint_{C_1}\frac{dz}{z^2e^z}$, (c) $\oint_{C_1}\frac{z^2dz}{\sinh 2z}$, (d) $\oint_{C_1}\left(\frac{z+1}{z-1}\right)^3 dz$, (e) $\oint_{C_2}\frac{dz}{z^2-2iz-2}$, (f) $\oint_{C_2}\frac{dz}{\cosh^2(\pi z/2)}$,

where C_1 is a closed rectangular contour, traversed anticlockwise, with vertices at $-1-i$, $3-i$, $3+3i$, $-1+3i$ and C_2 is a closed triangular contour, traversed clockwise, with vertices at -2, 2, $-2+3i$.

Ans.: (a) 0, (b) $2\pi i$, (c) $\pi^3 i/4$, (d) $12\pi i$, (e) πi, (f) 0.

6. Evaluate by means of the residue theorem.

(a) $\int_0^\infty\frac{dx}{x^4+a^4}$, (b) $\int_0^\infty\frac{dx}{(x^2+a^2)(x^2+b^2)}$ $a>0, b>0$, (c) $\int_0^\infty\frac{x^2dx}{x^4+1}$, (d) $\int_0^\infty\frac{dx}{(x^2+1)^2}$, (e) $\int_{-\infty}^\infty\frac{dx}{4x^2+2x+1}$,

(f) $\int_0^\infty\frac{x^2dx}{x^6+1}$, (g) $\int_0^\infty\frac{\cos(2x)dx}{(x^2+1)^2}$, (h) $\int_0^\infty\frac{x\sin(x)dx}{x^4+16}$, (i) $\int_{-\infty}^\infty\frac{\cos(x)dx}{8x^2+12x+5}$, (j) $\int_{-\infty}^\infty\frac{\sin(3x)dx}{2x^2+2x+1}$,

(k) $\int_0^\infty\frac{x^{a-1}dx}{x+1}$ $0<a<1$, (l) $\int_0^\infty\frac{\sqrt{x}dx}{x^3+1}$, (m) $\int_0^\infty\frac{dx}{\sqrt{x}(x^2+1)^2}$, (n) $\int_0^\infty\frac{\ln(x)dx}{x^2+1}$, (o) $\int_0^\infty\frac{x^{1/3}dx}{x^2+4}$,

(p) $\int_0^\infty e^{-x^2}\cos(2ax)dx$ $a>0$ (Hint: integrate around a rectangle with vertices at 0, R, $R+ia$, ia and use the known integral $\int_0^\infty e^{-x^2}dx=\sqrt{\pi}/2$), (q) $\int_0^\infty\frac{dx}{x^2+x+1}$ (Hint: introduce $\log z$ into the integrand),

(r) $\int_0^\infty\frac{dx}{x^3+1}$, (s) $\int_0^\infty\frac{dx}{(x^2+2x+10)^2}$, (t) $\int_0^\infty\frac{x\,dx}{x^3+1}$, (u) $\int_0^1\frac{dx}{x^2+1}$ (Hint: set $t=(1-x)/x$), (v) $\int_0^\infty\frac{\sin(x)dx}{x}$

Ans.: (a) $\pi/2\sqrt{2}a^4$, (b) $\pi/2ab(a+b)$, (c) $\pi\sqrt{2}/4$, (d) $\pi/4$, (e) $\pi/\sqrt{3}$, (f) $\pi/6$, (g) $3\pi/4e^2$, (h) $\frac{\pi e^{-\sqrt{2}}}{8}\sin\sqrt{2}$, (i) $\frac{\pi}{2}e^{-1/4}\cos\frac{3}{4}$, (k) $\pi/\sin a\pi$, (l) $\pi/3$, (m) $3\pi/4\sqrt{2}$, (n) 0, (p) $\frac{\sqrt{\pi}}{2}e^{-a^2}$, (q) $2\pi/3\sqrt{3}$, (r) $2\pi/3\sqrt{3}$, (s) $\frac{\pi}{108}-\frac{1}{180}-\frac{1}{54}\tan^{-1}(\frac{1}{3})$, (t) $2\pi/3\sqrt{3}$, (u) $\pi/4$, (v) $\pi/2$.

7. Evaluate by means of the residue theorem.

(a) $\int_0^{\pi} \sin^2(\theta)\,d\theta$, (b) $\int_0^{\pi} \cos^2(\theta)\,d\theta$, (c) $\int_0^{\pi/2} \sin^2(\theta)\,d\theta$, (d) $\int_{\pi/2}^{\pi} \cos^2(\theta)\,d\theta$, (e) $\int_0^{\pi} \sin^4(\theta)\,d\theta$,

(f) $\int_0^{\pi} \cos^4(\theta)\,d\theta$, (g) $\int_0^{\pi} \sin^6(\theta)\,d\theta$, (h) $\int_0^{4\pi} \cos^6(\theta)\,d\theta$, (i) $\int_{-\pi}^{\pi} \frac{d\theta}{7+\cos(\theta)}$, (j) $\int_0^{\pi} \frac{d\theta}{1+\cos^2(\theta)}$,

(k) $\int_0^{2\pi} \frac{d\theta}{1+\sin^2(\theta)}$, (l) $\int_0^{2\pi} \frac{d\theta}{2+\sin(2\theta)}$, (m) $\int_0^{\pi} \frac{\cos(\theta)\,d\theta}{1-2a\cos(\theta)+a^2}$ $-1<a<1$

Ans.: (a) $\frac{\pi}{2}$, (b) $\frac{\pi}{2}$, (c) $\frac{\pi}{4}$, (d) $\frac{\pi}{4}$, (e) $\frac{3\pi}{8}$, (g) $\frac{5\pi}{16}$, (i) $\frac{\pi\sqrt{3}}{6}$, (j) $\frac{\pi}{\sqrt{2}}$, (m) $\frac{\pi a}{1-a^2}$.

8. Use the inversion formula and the residue theorem to evaluate the inverse of the Laplace transforms, where $a>0$ and $b>0$.

   (a) $\frac{1}{s^2}$, (b) $\frac{1}{s^6}$, (c) $\frac{1}{s^2+a^2}$, (d) $\frac{1}{s^2-a^2}$, (e) $\frac{1}{(s-a)^4}$, (f) $\frac{1}{(s^2+a^2)^2}$, (g) $\frac{1}{(s-a)^2+b^2}$, (h) $\frac{e^{-as}}{s^3}$, (i) $\frac{e^{-as}}{(s-b)^2}$.

   Ans.: (a) t, (b) $t^5/5!$, (c) $\sin at/a$, (d) $\sinh at/a$, (e) $t^3e^{at}/6$, (h) $u_a(t)\frac{(t-a)^2}{2}$.

9. Use the inversion formula and the residue theorem to evaluate the inverse of the Fourier transforms.

   (a) $\frac{1}{\omega^2+i\omega+2}$, (b) $\frac{1}{\omega^2-3i\omega-2}$, (c) $\frac{1}{\omega^2+3i\omega-2}$, (d) $\frac{1}{(2-i\omega)^2}$, (e) $\frac{1}{(1+i\omega)^2}$, (f) $\frac{1}{(1+i\omega)^3}$, (g) $\frac{1}{(\omega^2+1)^2}$, (h) $\frac{1}{(\omega^2+4)^3}$.

   Ans.: (a) $e^{-(\sqrt{3}-1)x/2}/\sqrt{3}, x>0$; $e^{(\sqrt{3}+1)x/2}/\sqrt{3}, x<0$, (b) $-e^{-x}+e^{-2x}, x>0$; $0, x<0$, (c) $0, x>0$; $-e^{x}+e^{2x}, x<0$, (d) $0, x>0$; $-xe^{2x}, x<0$, (g) $\frac{1+|x|}{4}e^{-|x|}$.