EXERCISE SET #2

Definition: Let (x_0, y_0) be an interior point of the domain of definition D of a function $u = u(x,y)$. The limit of $u = u(x,y)$ as (x,y) approach (x_0, y_0) is said to be L,

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = L,$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that,

$$\text{if } (x,y) \in D \text{ and } \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta, \text{ then } |u(x,y) - L| < \varepsilon.$$

Definition: A function $u = u(x,y)$ is called continuous at (x_0, y_0), if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u(x_0, y_0).$$

Here the limit value needs to be independent of the way in which (x,y) approaches (x_0, y_0).

Definition: A function $u = u(x,y)$ is differentible at (x_0, y_0), if $\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$ can be expressed in the form

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Theorem: If the partial derivatives u_x and u_y exist near (x_0, y_0) and are continuous at (x_0, y_0), then u is differentible at (x_0, y_0).

Definition: The derivative of f at z_0 is defined as

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided that the limit exists and z_0 is an interior point of the domain of definition of f. Here the limit value needs to be independent of the way in which z approaches z_0.

Definition: $f(z) = u(x,y) + iv(x,y)$ is said to satisfy the Cauchy-Riemann equations if

$$u_x = v_y \text{ and } u_y = -v_x.$$

Differentibility: Let $f(z) = u(x,y) + iv(x,y)$ be defined throughout some neighborhood of a point $z_0 = x_0 + iy_0$. For f to be differentible at z_0,

(i) it is necessary that the Cauchy-Riemann equations be satisfied at x_0, y_0;

(ii) it is sufficient that the Cauchy-Riemann equations be satisfied at z_0, and that u and v be continuously differentible in some neighborhood of z_0.

If f is differentible, then f' is given by any of these four equivalent expressions:

$$f' = u_x + iv_x = u_y - iv_y = u_x - iu_y = v_y + iv_x$$

Definition: Suppose $f(z)$ is differentible at z_0 and throughout some neighborhood of z_0, then it is analytic at z_0. If it is not analytic at z_0, it is singular there. If it is analytic at each point of a region D, then it is analytic in D.

Definition: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v are harmonic in D, i.e. they are twice continuously differentible and they satisfy the Laplace equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad \nabla^2 v = v_{xx} + v_{yy} = 0$$

in D. Note that If $f(z) = u(x,y) + iv(x,y)$ is analytic, then the harmonic functions u and v are a related pair by Cauchy-Riemann equations. They are refereed to as conjugate harmonic functions..

Cauchy's Theorem: If f(z) is analytic in a simply connected domain D, then $\oint_C f(z)\,dz = 0$ for every piecewise smooth simple closed curve C in D.

Fundamental Theorem of the Complex Integral Calculus: Let f(z) be analytic in a simply connected domain D, and let z_0 be any fixed point in D. Then

(i) $G(z) = \int_{z_0}^{z} f(\xi)\,d\xi$ is analytic in D and $G'(z) = f(z)$.

(ii) If F(z) ia any primitive of f(z), i.e. $F'(z) = f(z)$, then $\int_{z_0}^{z} f(\xi)\,d\xi = F(z) - F(z_0)$.

Cauchy Integral Formula: Let $f(z)$ be analytic in a simply connected domain D, let C be a piecewise smooth simple closed curve in D oriented anticlockwise, and let z_0 be any point within C. Then

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0).$$

Furthermore,

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0).$$

1. Find the limit, if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y)\to(1,2)} \left(5x^3 - x^2y^2\right)$, (b) $\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)$, (c) $\lim_{(x,y)\to(2,1)} \frac{4-xy}{x^2+3y^2}$,

(d) $\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$, (e) $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$, (f) $\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$,

(g) $\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4}$, (h) $\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$, (i) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$, (j) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$,

(k) $\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4+4y^2}$, (l) $\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2}$, (m) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$,

(n) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

Ans.: (a) 1, (b) e, (c) 2/7, (d) 0, (e) DNE, (f) DNE, (g) DNE, (h) DNE, (i) 0, (j) 0, (k) DNE, (l) 0, (m) 2, (n) DNE.

2. Determine the set of points at which the function is continuous.

(a) $\frac{xy}{1+e^{x-y}}$, (b) $\cos\sqrt{1+x-y}$, (c) $\frac{1+x^2+y^2}{1-x^2-y^2}$, (d) $\frac{e^x+e^y}{e^{xy}-1}$, (e) $\ln\left(x^2+y^2-4\right)$, (f) $\tan^{-1}\left(\frac{1}{(x+y)^2}\right)$

(g) $\begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if} \quad (x,y)\neq(0,0) \\ 1 & \text{if} \quad (x,y)=(0,0) \end{cases}$, (h) $\begin{cases} \frac{xy}{x^2+xy+y^2} & \text{if} \quad (x,y)\neq(0,0) \\ 1 & \text{if} \quad (x,y)=(0,0) \end{cases}$.

Ans.: (a) everywhere, (b) $\{(x,y)|y\le x+1\}$, (c) $\{(x,y)|x^2+y^2\neq 1\}$, (d) $\{(x,y)|x\neq 0, y\neq 0\}$, (e) $\{(x,y)|x^2+y^2>4\}$, (f) $\{(x,y)|y\neq -x\}$, (g) $\{(x,y)|x\neq 0, y\neq 0\}$, (h) $\{(x,y)|x\neq 0, y\neq 0\}$.

3. Explain why the function is differentiable at the given point.

(a) $1+x\ln(xy-5)$, $(2,3)$, (b) x^3y^4, $(1,1)$, (c) $\frac{x}{x+y}$, $(2,1)$, (d) $\sqrt{x+e^{4y}}$; $(3,0)$.

Ans.: The function is differentible for (a) $\{(x,y)|xy>5\}$, (b) everywhere, (c) $\{(x,y)|y\neq -x\}$, (d) Both ∂_x and ∂_y are continuous near $(3,0)$,

4. Show that the function is differentiable by finding ε_1 and ε_2 that satisfy the definition of differentiability. (a) x^2+y^2, (b) $xy-5y^2$.

Ans.: (a) $\varepsilon_1=\Delta x$, $\varepsilon_2=\Delta y$, (b) $\varepsilon_1=\Delta y$, $\varepsilon_2=-5\Delta y$.

5. Determine where $f(z)=(x+\alpha y)^2+2i(x-\alpha y)$ is analytic for real constant α.

6. Determine $f'(z)$, where it exists, and state where f is analytic and where it is not.

(a) $(1-2z^3)^5$, (b) $(x+iy)/(x^2+y^2)$, (c) $|z|\sin z$, (d) $1/(z^2+3iz-2)$, (e) $1/(z^3+1)$,

(f) $x+i\sin y$, (g) z^{100}, (h) $\sqrt{z}$, (i) $1/\sqrt{z}$, both defined by the branch cut $(-\infty,0]$

Ans.: (a) everywhere, (b) nowhere, (c) differentiable at $z=n\pi,\ n=0,\pm1,\pm2$; analytic nowhere, (d) analytic everywhere except at $z=-i,-2i$, (e) analytic everywhere except at $z=-1,\frac{1}{2}(1\pm i\sqrt{3})$, (f) differentiable along the lines $y=\mp(2k+1)\frac{\pi}{2},\ k=0,1,2,\ldots$; analytic nowhere, (g) everywhere, (h) & (i) everywhere in the cut plane.

7. Let $f=u+iv$ be differentible with continuous partials at a point $z=re^{i\theta}$, $r\neq 0$. Show that the polar form of the Cauchy-Riemann equations is

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Use the polar form above to verify the analyticity of the function

$$f(z)=\frac{y^2-x^2}{(x^2+y^2)^2}+i\frac{2xy}{(x^2+y^2)^2}.$$

8. Show that the Cauchy-Riemann conditions (equations) are satisfied by

$$f(z)=\begin{cases} \dfrac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}, & z\neq 0 \\ 0, & z=0 \end{cases}$$

at $z=0$, but that f is not differentible at that point. Thus conclude that Cauchy-Riemann conditions are insufficient.

9. Let $C: z = z(t)$, $a \le t \le b$, be a regular arc and let z be a particular point on C and let $z + \Delta z = z(t + \Delta t) \in C$. The directional derivative of a function f, defined at least on C, at z in the direction of the arc is defined by

$$f'_C(z) = \lim_{\substack{\Delta z \to 0, \\ z+\Delta z \in C}} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided that the limit exists.

Show that $f'_C(z) = f_z + f_{\bar{z}} e^{-2i\theta}$ where $z'(t) = x'(t) + iy'(t) = |z'(t)| e^{i\theta}$. Thus conclude that if $f_{\bar{z}} \ne 0$ then the directional derivative of f at z depends on θ where the value $|f_{\bar{z}}|$ is called the deviation from analyticity. Here prime denotes differentiation wrt t.

Hint: Recall that $df = f_x dx + f_y dy$, $f_z = \frac{1}{2}(f_x - i f_y)$, and $f_{\bar{z}} = \frac{1}{2}(f_x + i f_y)$, so $df = f_z dz + f_{\bar{z}} d\bar{z}$. And that $f' = (f_z z' + f_{\bar{z}} \bar{z}')/z'$

10. Determine whether or not the given function u is harmonic and if so, in what region. If it is harmonic, find the most general conjugate function v and corresponding analytic function $f(z) = u + iv$ expressed in terms of z.

(a) $e^x \cos y$, (b) $e^{2x} \sin 2y$, (c) $x^3 - 3xy^2$, (d) $r^3 \sin 3\theta$, (e) $r^2 \cos 2\theta + 4$, (f) r,

(g) $x \cos 2x \cosh 2y + y \sin 2x \sinh 2y$, (h) $x^2 + y^2$.

Ans.: (a) $f(z) = e^z$, (b) $f(z) = -e^{2z}$, (c) $f(z) = z^3$, (d) $f(z) = -iz^3$.

11. Let C be the unit circular path centered at origin traversed in the anticlockwise direction in the complex plane. Show that for $0 \le k \le n$

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!} = \frac{1}{2\pi i} \int_C \frac{(1+z)^n}{z^{k+1}} dz .$$

12. Show that Green's theorem in the plane

$$\oint_C \{u\,dx + v\,dy\} = \iint_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$$

where u and v are two real-valued functions having continuous partial derivatives on a domain D containing an open set Ω and its boundary C, can be expressed by

$$\oint_C f(z)\,dz = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) dx\,dy$$

in the complex plane for $f(z) = u(x,y) + iv(x,y)$. Thus, show that

$$\oint_C \bar{z}\,dz = 2i \times \text{Area enclosed by } C .$$

13. Evaluate the following line integrals

(a) $\int_C |z|^2\, dz$; C is a straight line from 0 to $1+i$, (b) $\int_C \bar{z}\, dz$; C is the same as in (a), (c) $\int_C |z|^2\, dz$; C is a clockwise semicircle from 2 to -2 centered at 0, (d) $\int_C dz/z$; C consists of piecewise line segments from 1 to $1-i$, from $1-i$ to $-1-i$, from $-1-i$ to -1,
(e) $\int_C e^z dz$; C consists of piecewise line segments from i to $1+i$, from $1+i$ to $1-2i$,
(f) $\int_C \mathrm{Re}(z)\, dz$; C is a clockwise quarter circle from $3i$ to 3 centered at 0, (g) $\int_C \mathrm{Im}(z) dz$; C is a straight line from i to $2+2i$.

Ans.: (a) $\frac{1}{3}(2+2i)$, (b) 1, (c) $-4\pi i$, (d) $-\pi i$ √, (e) $(e\cos 2-\cos 1)-i(\sin 1+e\sin 2)$, (f) $\frac{1}{4}(18-9\pi i)$, (g) $\frac{1}{2}(6+3i)$.

14. Evaluate the following integrals using Cauchy's theorem, if applicable.

(a) $\int_{C_1} \mathrm{Re}(z) dz$, (b) $\int_{C_1} \mathrm{Im}(z) dz$, (c) $\int_{C_3} \mathrm{Im}(z) dz$, (d) $\int_{C_3} dz/(z^2-3)$, (e) $\int_{C_1} z^4 dz$,
(f) $\int_{C_1} dz/z(z-2)$, (g) $\int_{C_2} dz/z(z+5)$, (h) $\int_{C_1} e^{\sin z} dz$, (i) $\int_{C_2} \sin(\cos z) dz$, (j) $\int_{C_3} dz/|z|$,
(k) $\int_{C_1} \bar{z}\, dz$, (l) $\int_{C_3} \bar{z}\, dz$, (m) $\int_{C_4} dz/z(z-1)$, (n) $\int_{C_4} dz/z(z-5)$, (o) $\int_{C_4} z\, dz/(z^2+1)$,
(p) $\int_{C_4} z\, dz/(z^2-3z+2)$, (q) $\int_{C_4} dz/z^3(z^2-1)$, (r) $\int_{C_4} z\, dz/(z^3-1)$,

where $C_1 : |z|=1$; anticlockwise, $C_2 : |z|=1$; clockwise, C_3; the square with vertices at $1-i$, $1+i$, $-1+i$, $-1-i$ anticlockwise, $C_4 : |z|=3$; anticlockwise.

Ans.: (a) πi, (b) $-\pi$, (c) -4, (d) 0, (e) 0, (f) $-\pi i$, (g) $-\frac{2}{5}\pi i$, (h) 0, (i) 0, (j) 0, (k) $2\pi i$, (l) $8i$, (m) 0, (n) 0, (o) $2\pi i$, (p) $2\pi i$, (q) 0, (r) 0.

15. Use Cauchy Integral Formula to evaluate the following integrals.

(a) $\oint_C \frac{\cos z}{z} dz$, (b) $\oint_C \frac{\sin z}{z} dz$, (c) $\oint_C \frac{1}{z^2-5z} dz$, (d) $\oint_C \frac{z^2-1}{z^2+1} e^z\, dz$, (e) $\oint_C \frac{z+1}{(z-1)(z+2)^2} dz$,
(f) $\oint_C \frac{e^{2z}}{z^5} dz$, (g) $\oint_C \frac{\sinh 3z}{(z^2+1)^2} dz$, (h) $\oint_C \frac{z+2}{z^4-1} dz$, (i) $\oint_C \frac{e^{z^2}}{z\cos(z/2)} dz$, (j) $\oint_C \frac{z}{(z+i)(z^2+1)} dz$,
(k) $\oint_C \frac{z^3}{z^2+i} dz$, (l) $\oint_C \frac{1}{z(z-2)(z-4)} dz$

where $C : |z|=3$; anticlockwise.

Ans.: (a) $2\pi i$, (b) 0, (c) $-\frac{2}{5}\pi i$, (d) $-4\pi i \sin 1$, (e) 0, (f) $\frac{4}{3}\pi i$, (g) $\pi i(\sin 3-3\cos 3)$, (h) 0, (i) $2\pi i$, (j) 0, (k) 2π.

16. Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta .$$

expresses an analytic function $f(z)=u+iv$ in terms of its boundary values, thus it is expected there to exist a similar integral formula expressing a harmonic function $u(x,y)$ in terms of its boundary values. For this purpose, let C be the anticlockwise circle $|\zeta|=R$ or $\zeta = Re^{i\phi}$. Show that the Cauchy integral formula can be re-expressed as

$$f(z) = \frac{1}{2\pi i}\oint_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - R^2/\bar{z}}\right) f(\zeta)\, d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}}\right) f(\zeta)\, d\phi,$$

where the bracketed quantity is real. Show that for the circular disk $z = r e^{i\theta}$, it leads to the Poisson integral formula:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)}{R^2 - 2Rr\cos(\phi - \theta) + r^2}\, u(R,\phi)\, d\phi.$$