EXERCISE SET #1

1. Consider two complex numbers z_1 and z_2 in the first quadrant of the complex plane and corresponding vectors $\mathbf{u}$ and $\mathbf{v}$ in the x-y plane:
   (a) Show that the area of the parallelogram defined by sides z_1 and z_2 is $A = |\mathrm{Im}(\bar{z}_1 z_2)|$
   (b) Express the dot ($\mathbf{u} \cdot \mathbf{v}$) and the cross ($\mathbf{u} \times \mathbf{v}$) products in terms of z_1 and z_2.

2. Describe graphically the sets of points in the complex plane defined by the following equations or inequalities:
   (a) $\mathrm{Re}\, z \geq 0$, (b) $\mathrm{Im}\, z < 0$, (c) $|z| = 2$, (d) $|z-1| < 1$, (e) $z = 3 + (1+i)t, \quad t \in \mathbf{R}$,
   (f) $z = (1-t)i + t, \quad 0 \leq t \leq 1$, (g) $(z-i)(\bar{z}+i) = 4$, (h) $z + 1 = 3e^{it}, \quad 0 \leq t \leq 2\pi$,
   (i) $\mathrm{Arg}\, z = \frac{\pi}{4}$, (j) $\mathrm{Im}\frac{(z-2)}{i} > 0$, (k) $\{z: \quad |z| > 1, \quad |\mathrm{Arg}\, z| \leq \frac{\pi}{4}\}$, (l) $\mathrm{Re}\left(\frac{1}{z}\right) = 2$,
   (m) $|z+1| = |z-2|$, (n) $a < \mathrm{Re}\, z < b, \quad a, b \in \mathbf{R}$, (o) $2 \leq |z-i| \leq 3$, (p) $|z-1| \leq |z+1|$,
   (q) $|z-1| \leq 2|z+1|$, (r) $|z| - \mathrm{Re}\, z \leq \frac{1}{2}$, (s) $\mathrm{Im}\frac{(z+1)}{3i} = 0$, (t) $|z^2 - 1| = 1$,
   (u) $\mathrm{Arg}(z - z_0) = \alpha, \quad \text{const.}$, (v) $\mathrm{Im}\, z^2 = 1$, (w) $|z-1| + |z+1| = 4$, (x) $\mathrm{Im}\left(\frac{1}{z}\right) > \frac{1}{2}$,
   (y) $-\frac{\pi}{2} < \mathrm{Arg}\, z < \frac{\pi}{4}$, (z) $0 < \mathrm{Arg}\, z < \frac{3\pi}{4}$.

3. Write down in the form $z \to Az + B$ the following transformations of the complex plane:
   (a) translation in the direction $2-3i$.   (b) rotation about i through $\pi/4$.

4. If B lies on the circle with centre C and radius r, show that the equation of the tangent at B is $(\bar{B} - \bar{C})Z + (B - C)\bar{Z} = B\bar{B} - C\bar{C} + r^2$.

5. Show that the points 1, $i + 2$, $\frac{1}{5}(7 + 4i)$, all lie on a circle.

6. Show that the triangle with vertices at 0, z and w is equilateral if and only if $|z|^2 = |w|^2 = 2\,\mathrm{Re}(z\bar{w})$.

7. Find the value of each complex number below on the principal sheet of a multi-sheeted complex plane when the principal branch is defined by:
   (i) $-\pi < \theta \leq \pi$ (ii) $-\pi/2 < \theta \leq 3\pi/2$ (iii) $0 \leq \theta < 2\pi$
   (a) $\ln(3-4i)$, (b) $\ln(-3-4i)$, (c) $\ln(ie^{i})$, (d) $(3+4i)^{1/3}$, (e) $(i-1)^{1/4}$,
   (f) $(-ie^{-i})^{2/5}$

8. Determine the position(s) of the branch point(s) of each function below:

   (a) $(z-1)^{5/4}$, (b) $(z+2)^3/(z-2)^{4/5}$, (c) $((z+3)/(z-3))^{5/2}$, (d) $(z-1)^{3/2}(z+1)^2$,

   (e) $(z-1)^{1/2}(z-2)^{2/3}$, (f) $\ln(1/(4z+3))$, (e) $\ln((z^2-4z-5)/(z^3+z^2-z-1))$,

   (f) $(z^2-4)^{3/2}\ln(z)$

9. For each function below, determine the position(s) of the branch point(s) and the number of Riemann sheets required to make the function single-valued:

   (a) $(z-2)^3/(z+2)^{4/7}$, (b) $(z^2-4)^{3/7}$, (c) $(z+i)^{\pi}$, (d) $((z+4)/(z-4))^{5/3}$,

   (e) $(z+1)^2(z-1)^{3/2}$, (f) $(z^2-4)^{3/2}\ln(z)$, (g) $z^{1/2}+z^2$, (h) $z^{1/6}+z^{2/3}$,

   (f) $(z^2-1)^3\ln(z)$

10. Each function below has one or more branch points at just one value of z. For each function:

    - Identify that value of z.
    - Determine how many sheets are needed for the function to be single valued.
    - Extend the cuts from the branch point to $+\infty$ along the real axis and determine the values of the function along the two sides of the cut.
    - Extend the cuts from the branch point to $-\infty$ along the real axis and determine the values of the function along the two sides of the cut.

    (a) $(z-1)^{5/4}$, (b) $z^{1/2}z^{2/3}$, (c) $z^{3/5}\ln(z)$, (d) $(z-1)^{1/6}+(z-1)^{2/3}$, (e) $(z+2)^2+(z+2)^{2/3}$,

    (f) $z^{1/2}+\ln(z)$

11. Show that $(\sqrt{3}+i)^n+(\sqrt{3}-i)^n$ is real for any positive integer n.

12. Show (i) algebraically and (ii) geometrically that the equation $Az\bar{z}+\bar{B}z+B\bar{z}+C=0$, for real numbers A & C and complex number B represents a circline. The collective name circline stands for a circle or straight line in the complex plane.

13. Prove the identity $1+z+z^2+...+z^n=(1-z^{n+1})/(1-z)$ valis for all z, $z\neq 1$.

14. Let z_k be kth root of unity, i.e. kth root of $z^n=1$, for $k=1,2,...,n$. Show that $\sum_{k=1}^{n} z_k=0$.

15. Evaluate $(1+i\sqrt{3})^{(2-5i)}$.

16. Derive the formulas:

(a) $w = \sin^{-1} z = -i \log\left(iz + \sqrt{1 - z^2}\right)$, where $z = \sin w \equiv \frac{1}{2i}(e^{iw} - e^{-iw})$

(b) $w = \cos^{-1} z = -i \log\left(z + \sqrt{z^2 - 1}\right)$, where $z = \cos w \equiv \frac{1}{2}(e^{iw} + e^{-iw})$

(c) $w = \tan^{-1} z = -\frac{i}{2} \log\left((i - z)/(i + z)\right)$, where $z = \tan w \equiv \sin w / \cos w$

(d) $w = \cot^{-1} z = -\frac{i}{2} \log\left((z + i)/(z - i)\right)$, where $z = \cot w \equiv \cos w / \sin w$

(e) $w = \sinh^{-1} z = \log\left(z + \sqrt{1 + z^2}\right)$, where $z = \sinh w \equiv \frac{1}{2}(e^{w} - e^{-w})$

(f) $w = \cosh^{-1} z = \log\left(z + \sqrt{z^2 - 1}\right)$, where $z = \cosh w \equiv \frac{1}{2}(e^{w} + e^{-w})$

17. Complex velocity of the plane irrotational incompressible flow of a downward free stream U over a flat plate that extends from $x = -2a$ to $x = 2a$ is given by

$$u - iv = \frac{iUz}{\sqrt{z^2 - 4a^2}} .$$

The multivaluedness dictates using a branch cut extending from $-2a$ to $-2a$ along the real axis where the mathematical barrier presented by the cut corresponds to the physical barrier presented by the plate. Find the velocity components $u(x, 0\pm)$ and $v(x, 0\pm)$ on the top and bottom of the plate.

18. Determine the complex function $w = f(z) = u(x, y) + iv(x, y)$ from its components:

(a) $u(x, y) = x^3 - 3xy^2, \quad v(x, y) = 3x^2 y - y^3,$

(b) $u(x, y) = x^2 + y^2, \quad v(x, y) = xy,$

(c) $u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$