# 1. INTRODUCTION

The concept of stability in examining the dynamics of any physical system first requires the establishment of an equilibrium. In fact, stability is defined as “the ability of a dynamical system to be immune to small disturbances”. In the case of instability the disturbance becomes amplified and as a result there is a departure from any state of equilibrium the system had. Thus the definition implicitly refers to an equilibrium. If an equilibrium does not exist, the underlying system is already unstable.

Hydrodynamic stability involves the sudy of the stability in fluid mechanics. It is described by the Navier-Stokes (N-S) equations for the conservation of momentum and the conservation of mass constrained by, say, the incompressibility condition, and the boundary conditions. The desciption may involve the choice of a coordinate system, such as rectangular or curvilinear coordinates, and external forces such as gravity, electrical or magnetic forces. The important step in studying the hydrodynamic stability is to identify the equilibrium (base or mean) flow. A flow in equilibrium does not necessarily mean time-independent, but it is no longer accelerated due to the balance of all forces. The nonlinearity of the governing equations of motion allows only few exact equilibrium solutions in closed form (Drazin, 2006), the most are obtained by approximations.

Base or mean flows may be categorized in three groups:

**(1)** Flows that are parallel, such as channel flows, plane Coutte (driven by the sliding motion of the boundary) and Poiseuille (driven by a pressure gradient) flows, where the flows are confined by two solid boundaries. Pipe flow in a polar coordinate system is another example. There are also almost parallel flows such as (i) free-shear flows (jet, wake, mixing layer) where there is no solid boundaries and (ii) the flat plate boundary layer flow where there is one solid boundary. As opposed to one component of the mean flow in the case of parallel flows, almost parallel flows have two components $\boldsymbol{U}(x, y) = (U, V)$ as functions of two coordinates: one in the direction of the flow, $x$ and the other along the extent of the flow, $y$. Almost parallel implies that $V \ll U$ and that $U$ is weakly dependent on the downstream variable $x$.

**(2)** Flows with curved steamlines, such as, flow between concentric circular cylinders (Taylor problem) or flow on concave walls (Görtler problem).
**(3)** Flows where the mean flow has a zero value, such as, Raylegih-Benard thermal convection flow where the instability in the form of convective motions establishes itself over the conduction state where there is no motion.

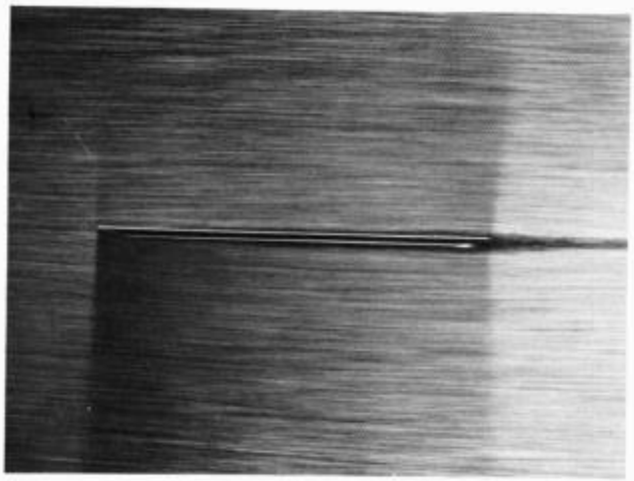

**Fig.1.1:** Laminar Boundary Layer

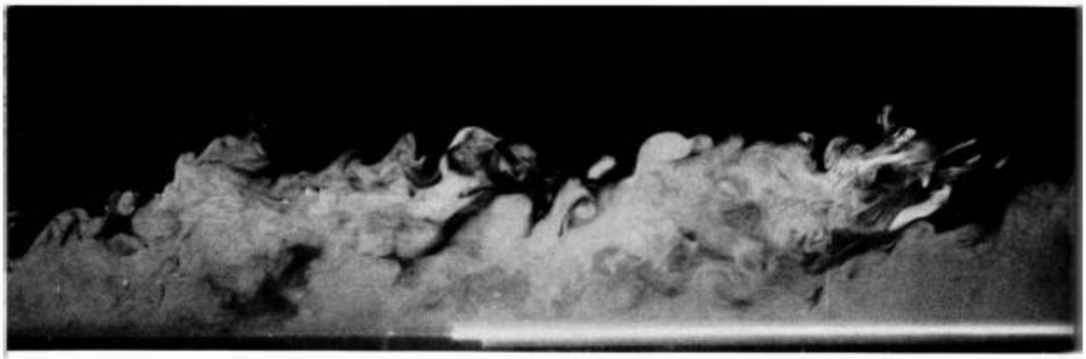

**Fig.1.2:** Turbulent Boundary Layer

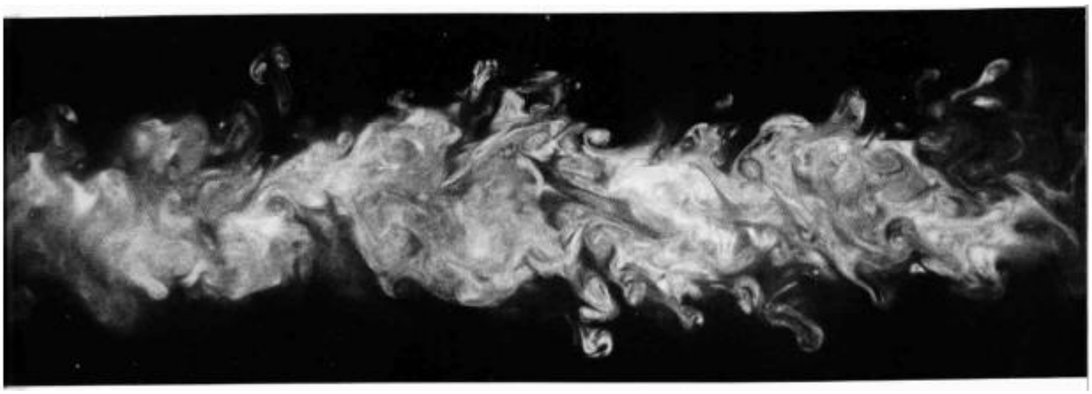

**Fig.1.3:** Turbulent Wake Flow

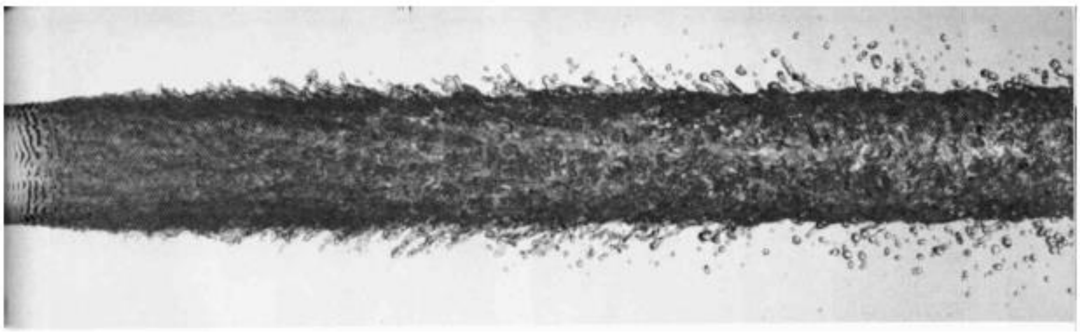

**Fig.1.4:** Jet Flow

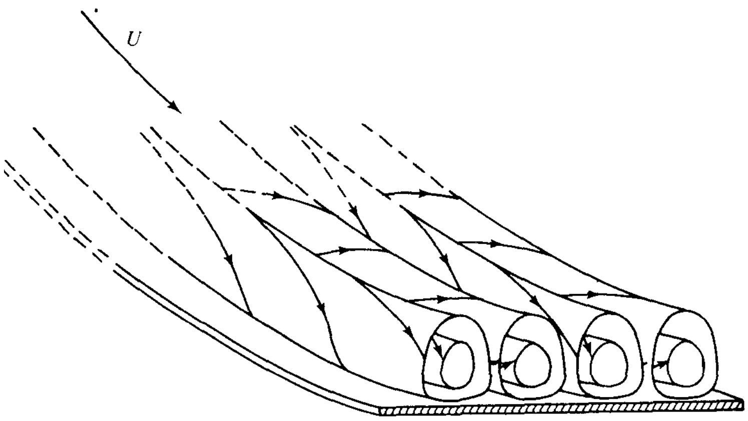


**Fig.1.5:** Görtler vortices

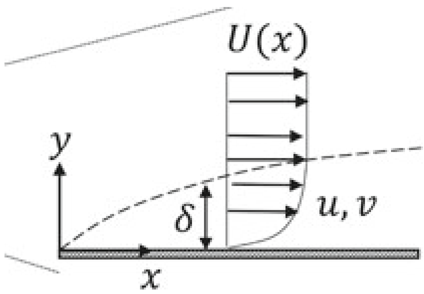


**Fig.1.6:** Boundary Layer over flat plate

The mean flows are laminar. The laminar state is extremely hard to maintain, the transition to turbulence eventually occurs in the short or in the long term. Laminar flows are orderly, predictable and mostly desired due to incurring less drag in comparison to turbulent flows. Main benefit of turbulent flows is rapid mixing. Stability analysis is an attempt to predict the process of transition from laminar to turbulent flow through the stages of bifurcations depending on some flow parameters, such as Reynolds number (Re), Rayleigh number (Ra), starting from the secondary flow superimposed over the primary (mean) flow. Görtler vortices in Fig.

1.5 are secondary flows superimposed over the mean flow on concave walls. There is a rich collection of actual photographs of laboratory flows in Van Dyke (1988).

The linear stability theory is able to explain why a basic flow can not be maintained indefinitely for most of the major cases. However, there are cases that linear stability theory is inconclusive, such as pipe flow. In the classical experiment by Reynolds (1883) for flow in a circular pipe flow, it was shown that the originally organized parallel laminar flow breaks down into fully random 3D flow, thus, clearly showing that the flow is unstable.

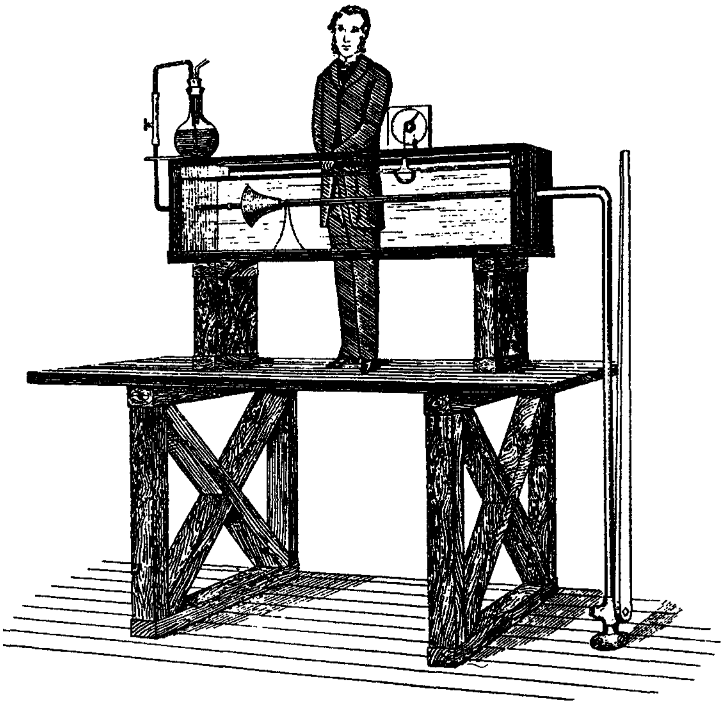

**Fig.1.7:** Boundary Layer over flat plate

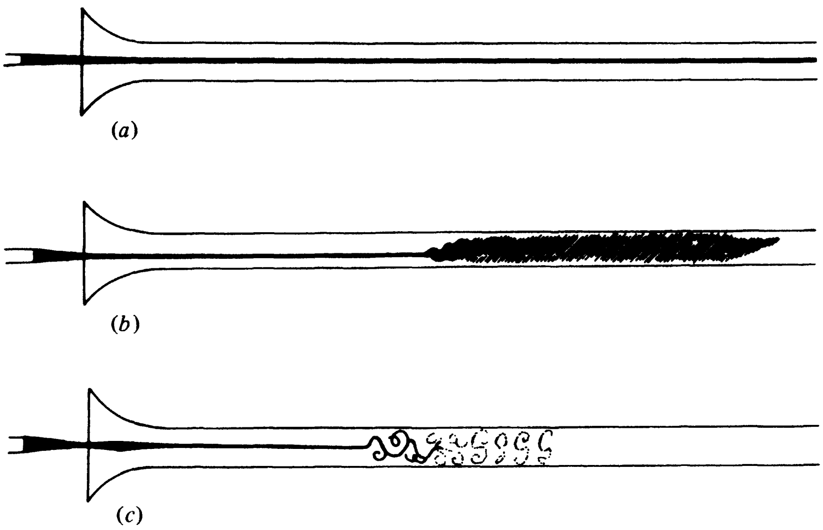


**Fig.1.8:** (a) Laminar flow in a pipe, (b) transition to turbulent flow, (c) the illuminated details of the transition.

Historically, Rayleigh (1883) and Kelvin (1879) were the pioneers in the field of hydrodynamic stability. They mainly worked on the problems under the inviscid approximation, unlike Reynolds who worked with viscous flow. Independently, Orr (1907) and Sommerfeld (1908) worked on the viscous stability problem to investigate the stability of channel flow. Their work led to the Orr-Sommerfeld equation that is fundamental in the theory of hydrodynamic stability. First computations on the Orr-Sommerfeld equation was done by Tollmien (1929), and later by Schlichting (1932) that led to Tollmien-Schlichting waves. These waves are fundamental in the stability of parallel or nearly parallel flows and appear in the stability study of viscous flows bounded by a solid boundary. Taylor (1923) developed the stability theory for the case of flow between rotating concentric cylinders. The theory was confirmed by the experiments and this was a big boost for the stability analysis.

The concept and types of stability can be summarized in the following illustration of a mechanical system that depicts an object at an equilibrium (base) state in various configurations. The stability of the object (or maintainability of its base state) can be tested by an infinitesimal disturbance. The state in (a) is termed as stable, (b) is unstable and (c) is neutral (indefinite). In (d), the state is stable to small and unstable to large disturbances. This is called conditional stability.

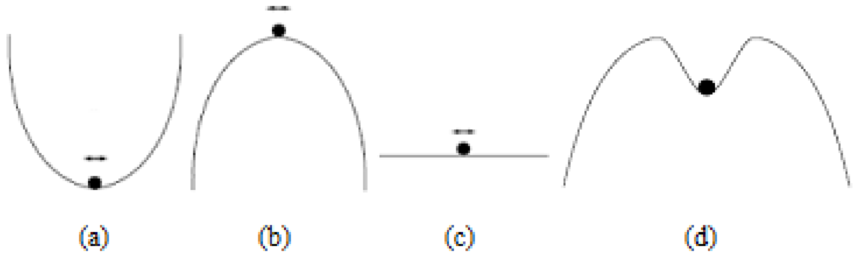


**Fig.1.9:** The state in (a) is stable, (b) unstable, (c) neutral and (d) conditionally stable.

Consider now a basic (laminar) flow characterized by the velocity field $\boldsymbol{U}(\boldsymbol{x}, t)$ that satisfy the N-S equations together with some boundary conditions. In order to test the stability the basic flow, it is disturbed by $\boldsymbol{u}(\boldsymbol{x}, t)$ and its evolution as governed by N-S equations is studied. The stability of the basic flow now can be defined as follows:

**<u>Definition</u>**: A basic flow is stable (in the sense of Liapounov), if for any $\varepsilon > 0$, there exists some positive number $\delta$ (depending upon $\varepsilon$ such that if

$$\text{if } \|\boldsymbol{u}(\boldsymbol{x}, 0) - \boldsymbol{U}(\boldsymbol{x}, 0)\| < \delta, \text{ then } \|\boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{U}(\boldsymbol{x}, t)\| < \varepsilon$$

for all $t > 0$. (see Fig. 10(a)).

Without loss of generality, the mean flow may be taken to have a zero value $\boldsymbol{U} = \boldsymbol{0}$, then this definition means that the flow is stable if the perturbation is small for all time provided it is small initially, or, if the solution is uniformly continuous for all time with respect to the initial conditions. Here the smallness and continuity is assigned by the norm, such as, max or infinity norm $\|\boldsymbol{u}(\boldsymbol{x}, t)\|_\infty \equiv \max_x |\boldsymbol{u}(\boldsymbol{x}, t)|$.

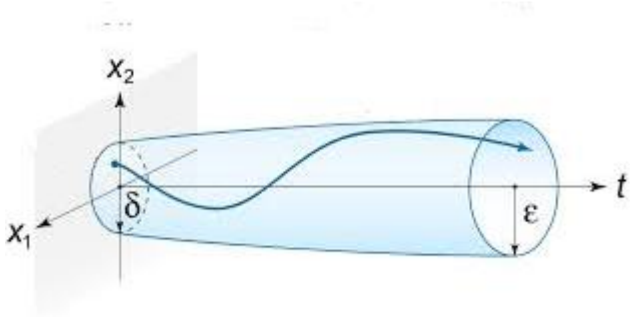


**Fig.1.10:** (a) Liapunov stability illustrated.

Similarly, it is asymptotically stable if $\|\boldsymbol{u}(\boldsymbol{x},t)\| \to 0$ as $t \to \infty$ (Fig. 10(b)).

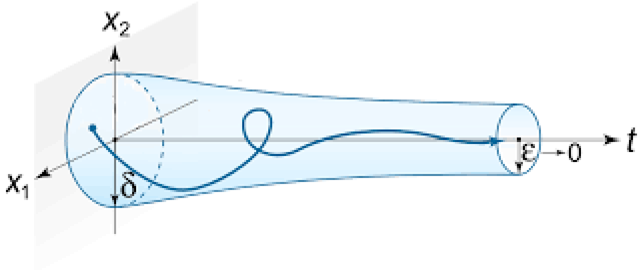


**Fig.1.10:** (b) Asymptotic stability illustrated.

If there is a value $\delta^* > 0$ such that it is stable when $\|\boldsymbol{u}(\boldsymbol{x},0)\| < \delta^*$, the basic flow is called **conditionally stable**. This implies that if $\|\boldsymbol{u}(\boldsymbol{x},0)\| \geq \delta^*$, the disturbance grows or forms a new stable state (exchange of stabilities). If $\delta^* \to \infty$, the basic state is **globally** or **unconditionally stable**.

If the flow is asymptotically stable and $\|\boldsymbol{u}(\boldsymbol{x},t)\| \leq \|\boldsymbol{u}(\boldsymbol{x},0)\|$ for all $t > 0$, then it is **monotonically stable**.

In Fig.1.11, for example, (a,b) shows monotonic (1) versus non-monotonic (2) stability, while (c,d) shows conditional stability. For initial conditions outside the gray triangular area the mean flow $\boldsymbol{U} = \boldsymbol{0}$ is unstable. Here, $\boldsymbol{u} = (\mathrm{u}, \mathrm{v})$ and $E = \|\boldsymbol{u}\|_2 = \left(\int |\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x}\right)^{1/2}$ is the energy norm.

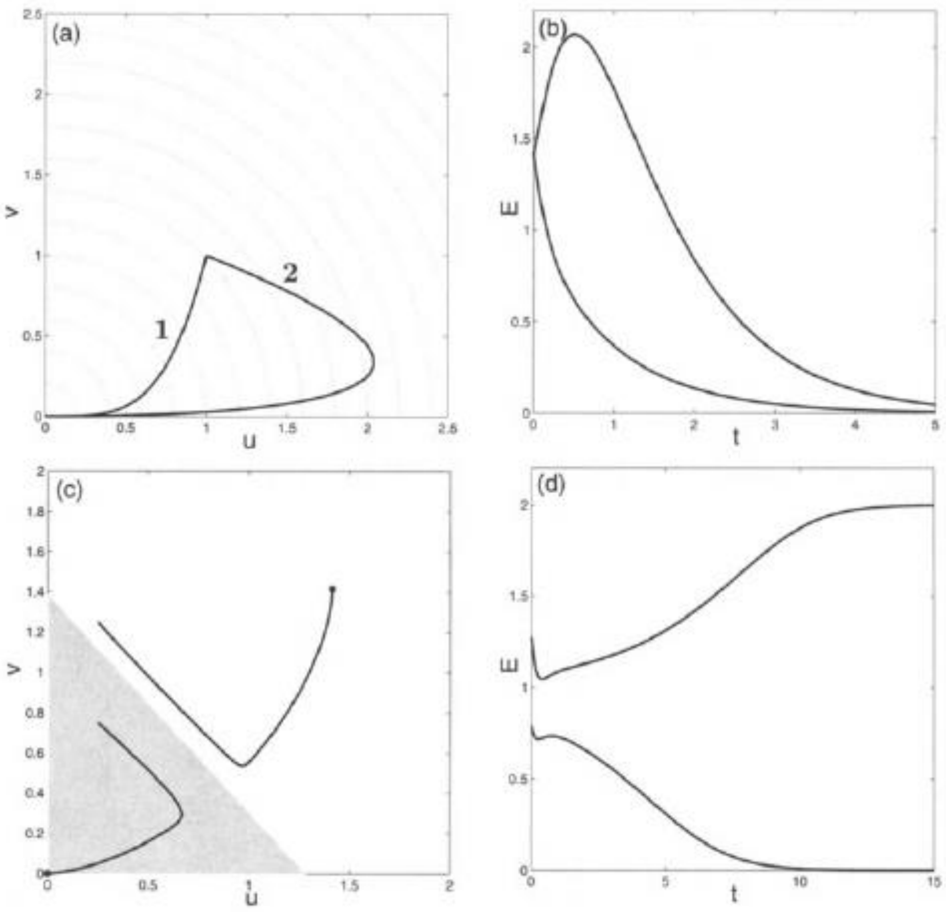


**Fig.1.11**: (a,b) Monotonic (1) versus non-monotonic (2) stability, (c,d) conditional stability.

Hence, different laminar-turbulent transition scenarios occur when the basic flow loses asymptotic, conditional, global or monotonic stability. The passage to turbulent flow through the instabilities is controlled by dimensionless problem parameters, such as Reynolds number (Re), the ratio between inertial and friction forces. Based on the stability definitions given above, the critical Re separating the regions of stable and unstable states may be illustrated in Fig. 1.12:

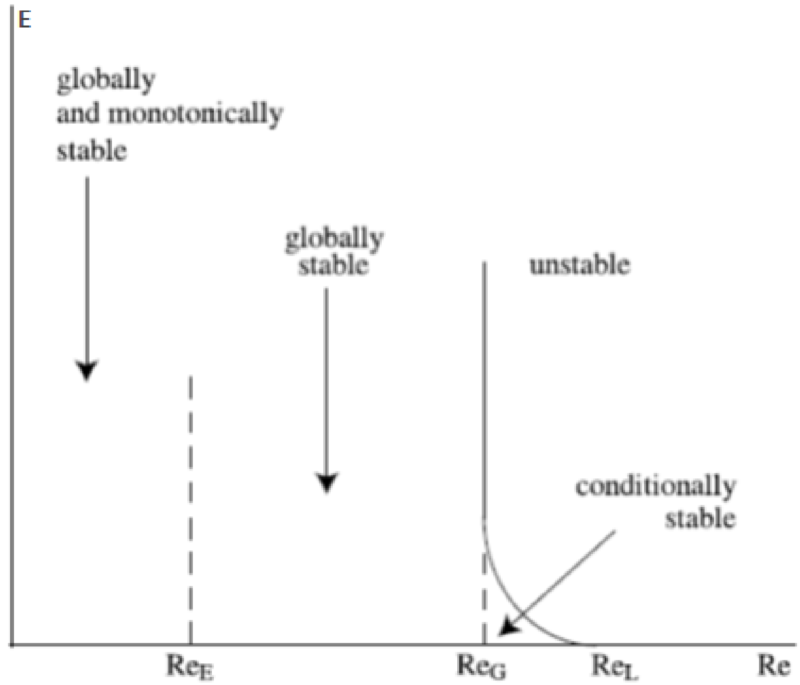


**Fig.1.12**: (a,b) Monotonic (1) versus non-monotonic (2) stability, (c,d) conditional stability.

As it is known the transition to turbulence occurs at high Re, while flows with relatively low Re are always laminar. In this picture, it is conjectured that the less restricitve is the definition of instability, the higher is Re corresponding to it.At $Re \geq \mathrm{Re_L}$, the flow is linearly unstable, i.e. there is an infinitesimal disturbance that does not decay in time.

Different types of disturbance evolution are shown below. The top row describes the evolution of disturbances that are localized in space, whereas the middle row illustrates the evolution of an unstable wavelike disturbance in time. The lower row depicts an oscillatory disturbance localized in space that exhibits (spatial) growth as it propagates downstream.

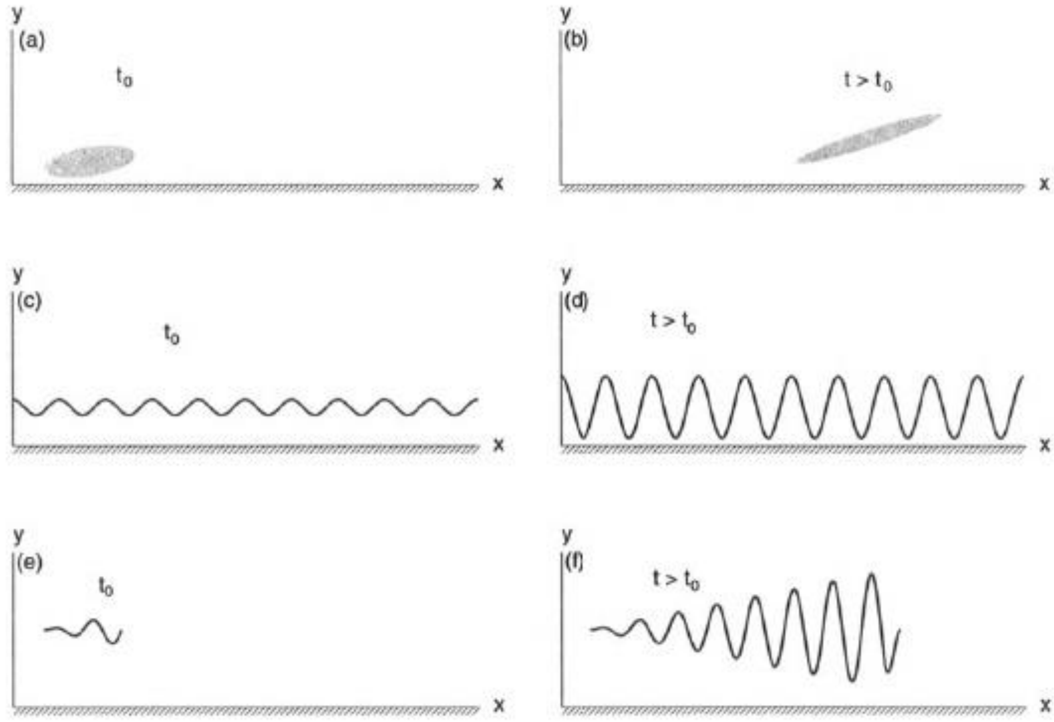


**Fig.1.13**: (a,b) Evolution of the localized disturbance in space, (c,d) evolution of an unstable wavelike disturbance in time, and (e,f) evolution of oscillatory disturbance localized in space that exhibits (spatial) growth as it propagates downstream.

An alternative illustration of the evolution types may be given as in Fig. 1.14:

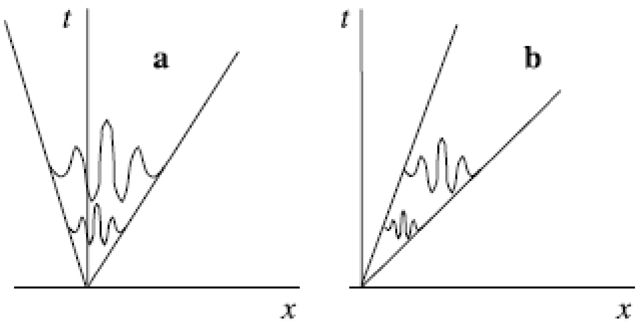


**Fig.1.14:** (a) Absolutely unstable and (b) convectively unstable disturbances.

In (a), it is the instability in time. The initial disturbance grows in time at a fixed point x, that is the system is absolutely unstable. In (b), it is instability in space. An initial signal is amplified whilst propagating, that is the system is convectively unstable.

A caricature model of conditional stability is provided by Landau equation (Landau & Lifshitz, 1959):

$$\frac{d|A|^2}{dt} = \lambda_1|A|^2 + \lambda_2|A|^4$$

where A is the amplitude of the disturbance and the constants $\lambda_1$ and $\lambda_2$ are the Landau coefficients which can be either positive or negative and are specific to the particular flow under consideration. The Landau equation is derived in the neighborhood of the neutral curve, i.e. $|Re - Re_L| \ll 1$, by a multiple-scale analysis of the equations of motion (Drazin & Reid, 1981).

The equation can be rewritten as

$$\frac{d|A|^{-2}}{dt} = -\lambda_1|A|^{-2} - \lambda_2$$

in order to obtain the general solution

$$|A|^{-2}(t) = |A_0|^{-2}exp(-\lambda_1 t) - \frac{\lambda_2}{\lambda_1}\big(1 - exp(-\lambda_1 t)\big)$$

or

$$|A|^2(t) = \frac{|A_0|^2 exp(\lambda_1 t)}{1 + \frac{\lambda_2}{\lambda_1}|A_0|^2\big(1 - exp(\lambda_1 t)\big)}$$

where $A_0$ is the initial value. Clearly, $\lambda_1 > 0$ when $Re - Re_L > 0$ and , $\lambda_1 < 0$ when $Re - Re_L < 0$. Here are some possibilities depending on the sign of the coefficients:

1. $\lambda_1 > 0,\ \lambda_2 < 0$ : The solution gives steady state at $A_f = \sqrt{-\lambda_1/\lambda_2}$ as $t \to \infty$ independent of $A_0$, i.e. the exchange of stabilities occurs.
2. $\lambda_1 < 0, \lambda_2 < 0$ : $A_f \to 0$, i.e. the flow is stable.
3. $\lambda_1 > 0, \lambda_2 > 0$ : Unbounded growth occurs independent of $A_0$. The solution becomes infinite at $t = \ln(1 + \lambda_1/(\lambda_2|A_0|^2))/\lambda_1$. It can be considered as fast transition to turbulence.
4. $\lambda_1 < 0,\ \lambda_2 > 0$ : If $A_0 < \sqrt{-\lambda_1/\lambda_2}$, then $A_f \to 0$, i.e. the flow becomes stable. If $A_0 > \sqrt{-\lambda_1/\lambda_2}$, then unbounded growth occurs and the solution becomes infinite at $t = \ln(|A_0|^2/(|A_0|^2 - |A_f|^2))/\lambda_1$.

Thus, two types of instability phenomena can be observed. As in cases (1) and (3), a nonzero steady state or turbulence is reached by introducing an infinitesimal disturbance at $Re > Re_L$ (supercritical regime) (Fig. 1.15(a)) that the Blasius boundary layer flow is an example. Or as in case (4), flow instability can be triggered at $Re < Re_L$ (subcritical regime) with an initial disturbance of finite amplitude that the basic motion ($A = 0$) at this subcritical regime is metastable (Fig. 1.15(b)) that the plane Couette flow and plane Poiseuille flows are some examples.

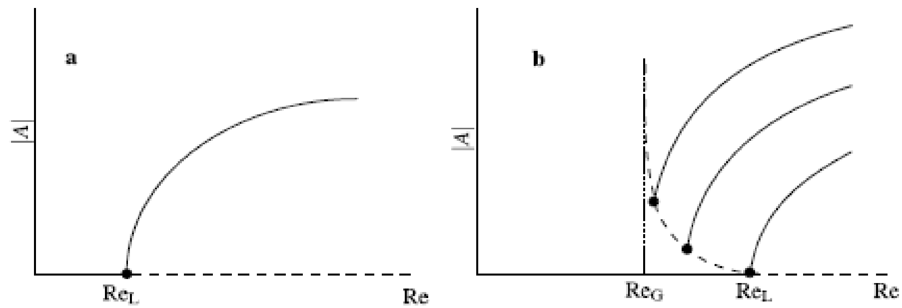


**Fig.1.15:** (a) For $Re > Re_L$ (supercritical regime), the basic motion ($A = 0$) loses stability (dash line) to a new state (solid line) while (b) for $Re_G < Re < Re_L$ (subcritical regime), the basic motion is metastable and finite disturbances trigger passage to a new state.

(Linear) Stability theory uses (infinitesimal) perturbations to test the stability of the mean flow. Consider for example a mean flow $\boldsymbol{U}$ with mean pressure $P$ characterizing the base flow state. Assume that the mean flow is disturbed by a fully 3D disturbance

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'(x, y, z, t); \qquad p = P + p'$$

and its evolution under the governing incompressible N-S equations are sought

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{\boldsymbol{u} \cdot \nabla \boldsymbol{u}}_{N\{\boldsymbol{u},\boldsymbol{u}\}} = \underbrace{-\nabla p + Re^{-1}\nabla^2 \boldsymbol{u}}_{L\{\boldsymbol{u},p;Re\}}$$

where $N\{\boldsymbol{u}, \boldsymbol{u}\}$ represents the nonlinear term and $L\{\boldsymbol{u}, p; Re\}$ represents the linear terms. Substituting the disturbed flow into N-S equations yield

$$\nabla \cdot (\boldsymbol{U} + \boldsymbol{u}') = 0,$$

$$\frac{\partial(\boldsymbol{U} + \boldsymbol{u}')}{\partial t} + N\{\boldsymbol{U} + \boldsymbol{u}', \boldsymbol{U} + \boldsymbol{u}'\} = L\{\boldsymbol{U} + \boldsymbol{u}', P + p'; Re\}.$$

Since the base flow already satisfies the N-S equations, it reduces to

$$\nabla \cdot \boldsymbol{u}' = 0,$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + N\{\boldsymbol{U}, \boldsymbol{u}'\} + N\{\boldsymbol{u}', \boldsymbol{U}\} - L\{\boldsymbol{u}', p'; Re\} = -N\{\boldsymbol{u}', \boldsymbol{u}'\}.$$

These equations are linearized by ignoring $N\{\boldsymbol{u}', \boldsymbol{u}'\}$ term owing to the assumption of infinitesimal disturbances and then used to study the evolution of the infinitesimal disturbances.

Consider a parallel (or almost parallel) flow with the mean

$$\boldsymbol{U} = (U(y), 0, 0)$$

where the mean flow is in x-direction and varies in the y-direction, z is in the transverse direction. In channel flow, for example, x and z range from minus to plus infinity, while y defines the solid boundaries. The linearized disturbance equations involves coefficients as functions of y only due to $U(y)$ appearing in $N\{\boldsymbol{U}, \boldsymbol{u}'\}$ and $N\{\boldsymbol{u}', \boldsymbol{U}\}$ terms. The disturbance can then be Fourier transformed

$$\widehat{\boldsymbol{u}}'(\alpha, y, \gamma, t) = \iint_{-\infty}^{+\infty} \boldsymbol{u}'(x, y, z, t) \exp(i(\,\alpha x + \gamma z)) dx dz$$

and similarly for $p'$ where $\alpha$ and $\gamma$ are real transform variables. Since the disturbance equations are linear, it can further be reduced to an ODE by applying Laplace transform in time. Classically, however, it is assumed that time dependence can be separated as follows

$$\widehat{\boldsymbol{u}}'(\alpha, y, \gamma, t) = \sum_{n=0}^{\infty} \widetilde{\boldsymbol{u}}'(\alpha, y, \gamma) \exp(-i\omega_n t)$$

where $\omega_n$ is taken as a complex frequency with its imaginary part rendering the corresponding mode indexed by n as unstable mode when positive. The substitution of these representations into the disturbance equations result in a differential eigen problem owing to the homogeneity of the boundary conditions in y as well. The eigenvalues $\omega_n$ are now functions of $\alpha$, $\gamma$ and Re. This is classical normal mode analysis.

The experimental studies in the investigations of stability in flat plate boundary layers, however, are based on the downstream measurements on a disturbance introduced at an initial x-location upstream on the flat plate (see Fig. 1.13 (a,b)). As opposed to the previous case of a temporal initial-value problem, this is actually a spatial initial-value problem and the resulting formulation should reflect this. The behavior in time is simply taken as periodic, so $\omega$ must be purely real. The wave numbers $\alpha$ and $\gamma$ are to be complex. The disturbance evolution is now posed as a spatial initial-value problem with the initial values are functions of y and z, and set at $\alpha$, $x = x_0$. In order to satisfy the far field boundary conditions as $|z| \rightarrow \infty$, $\gamma$ is taken as real. Thus another eigen problem results for the eigenvalue $\alpha$ as function of $\omega$, $\gamma$ and Re. Negative imaginary part of $\alpha$ indicates instability. This constitutes a spatial stability analysis as opposed to the earlier temporal stability analysis.

## REFERENCES


Drazin, P. G. and Riley, N. (1981). Hydrodynamic Stability. Cambridge University Press.

Drazin, P. G. and Riley, N. (2006). The Navier-Stokes Equations: A Classification of Flows and Exact Solutions. Cambridge University Press.

Landau, L. D. and Lifshitz, E. M. (1959). Fluid Mechanics. Vol. 6 of Course of Theoretical Physics, Pergamon Press.

Van Dyke, M. (1988). An Album of Fluid Motions, Parabolic Press.