SOME FORMULAS

- <u>Frobenius Theorem</u>: Let $x^2y''+xp(x)y'+q(x)y=0$ be a 2^{nd} order homogeneous linear differential equation whose coefficients are analytic in the interval $|x|<R$. Then it has two linearly independent solutions:
  - For r_1-r_2 not an integer: $y_1(x)=|x|^{r_1}\sum_{k=0}^{\infty}a_kx^k$, $y_2(x)=|x|^{r_2}\sum_{k=0}^{\infty}b_kx^k$.
  - For $r_1=r_2=r$: $y_1(x)=|x|^{r}\sum_{k=0}^{\infty}a_kx^k$, $y_2(x)=|x|^{r}\sum_{k=1}^{\infty}b_kx^k+y_1(x)\ln|x|$.
  - For r_1-r_2 positive integer: $y_1(x)=|x|^{r_1}\sum_{k=0}^{\infty}a_kx^k$, $y_2(x)=|x|^{r_2}\sum_{k=0}^{\infty}b_kx^k+cy_1(x)\ln|x|$,

  where r_1 and r_2 are the indicial roots and c is a constant.

- Let $F(s)=L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)dt$ be the Laplace Transform of $f(t)$:
  - $L\{e^{at}\}=\frac{1}{s-a}$ for $s>a$
  - $L\{t^nf(t)\}=(-1)^n\frac{d^n}{ds^n}F(s)$
  - $L\{\cos(at)\}=\frac{s}{s^2+a^2}$
  - $L\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\}=F(s)G(s)$
  - $L\{\sin(at)\}=\frac{a}{s^2+a^2}$
  - $L\{t^n\}=\frac{n!}{s^{n+1}}$ for n nonnegative integer
  - $L\{e^{at}f(t)\}=F(s-a)$
  - $L\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$ for f periodic with period T
  - $L\{f(t-a)u(t-a)\}=e^{-as}F(s)$ where u is the unit step function
  - $L\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-...-f^{(n-1)}(0)$
  - $L\{\delta(t-a)\}=e^{-as}$ where δ is the Dirac delta function modelling impulsive force $g(t)$ acting on extremely short time interval $a\le t\le a+\tau$, and $\int_a^{a+\tau}g(t)dt=1$.

- Let $y_1(t)$ and $y_2(t)$ be two solutions of the linear differential equation $y''+p(t)y'+q(t)y=0$. Then, their Wronskian $W(t)=W[y_1,y_2](t)=y_1(t)y_2'(t)-y_1'(t)y_2(t)$ satisfies the first-order differential equation $W'+p(t)W=0$

- Let $y_1(t)$ and $y_2(t)$ be two solutions of the linear differential equation $y''+p(t)y'+q(t)y=0$. Then, the following form $y(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ leads to a particular solution of the nonhomogeneous equation $y''+p(t)y'+q(t)y=f(t)$ for $u_1'(t)=-\frac{f(t)y_2(t)}{W(t)}$ and $u_2'(t)=\frac{f(t)y_1(t)}{W(t)}$.

- Miscellaneous power series expansions:
  - $\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k$ for $|x|<1$
  - $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$
  - $e^{ix}=\sum_{k=0}^{\infty}\frac{(ix)^k}{k!}=\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k}}{2k!}+i\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k+1}}{(2k+1)!}=\cos(x)+i\sin(x)$

- (Regular) Sturm-Liouville (S-L) problem is a linear homogeneous 2nd order boundary value problem (BVP): $[p(x)y']'+q(x)y+\lambda w(x)y=0$, in (a,b), with homogeneous boundary condition $\alpha y(a)+\beta y'(a)=0$, $\gamma y(b)+\delta y'(b)=0$, where a, b are finite, where p,p',q,w are continuous on $[a,b]$, and where $p(x)>0$ and $w(x)>0$ on $[a,b]$. Further, α,β are not both zero, γ,δ are not both zero, and a,b, $p(x),q(x),w(x)$, $\alpha,\beta,\gamma,\delta$ are all real.

- (Periodic) S-L problem has the nonseparated (periodic) boundary conditions: $y(a)=y(b)$, $y'(a)=y'(b)$.

- (Singular) S-L problem arises when $p(x)$ (and possibly $w(x)$) vanishes at one or both endpoints, so that $p(x)>0$ and $w(x)>0$ holds on open (a,b). Further, the boundary conditions are modified as follows:

  (1) $p(a)=0$ (and $p(b)\neq 0$): Then the boundary conditions are: y bounded at a, $\gamma y(b)+\delta y'(b)=0$.

  (2) $p(b)=0$ (and $p(a)\neq 0$): Then the boundary conditions are: $\alpha y(a)+\beta y'(a)=0$, y bounded at b.

  (3) $p(a)=p(b)=0$: Then the boundary conditions are: y bounded at a, y bounded at b.