EXERCISE SET #3

Definition: (Regular) Sturm-Liouville (S-L) problem is a linear homogeneous 2nd order boundary value problem (BVP):

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0, \qquad a < x < b,$$

with homogeneous (separated) boundary conditions

$$\alpha y(a) + \beta y'(a) = 0, \qquad \gamma y(b) + \delta y'(b) = 0,$$

where a, b are finite, where p, p', q, w are continuous on $[a,b]$, and where $p(x) > 0$ and $w(x) > 0$ on $[a,b]$. Further, α, β are not both zero, γ, δ are not both zero, and a,b, $p(x), q(x), w(x)$, $\alpha, \beta, \gamma, \delta$ are all real.

Theorem: (S-L Theorem) Let λ_n and $\phi_n(x)$ denote any eigenvalue and corresponding eigenfunction of the S-L eigenvalue problem $L[y] = \lambda y$ where L is the (S-L) differential operator

$$L \equiv -\frac{1}{w}\left[\frac{d}{dx}\left(p\frac{d}{dx}\right) + q\right].$$

(a) The eigenvalues are real,

(b) To each eigenvalue there corresponds only one linearly independent eigenfunction. Further there are an ∞-number of eigenvalues and they can be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ where $\lambda_n \to \infty$ as $n \to \infty$.

(c) Eigenfunction corresponding to distinct eigenvalues are orthogonal, that is, if $\lambda_j \neq \lambda_k$, then $\langle \phi_j, \phi_k \rangle_w = 0$, where the weighted "inner" product is defined by $\langle \phi_j, \phi_k \rangle_w \equiv \int_a^b \phi_j(x)\bar{\phi}_k(x)\, w(x)dx$.

(d) Let f and f' be piecewise continuous on $[a,b]$. If $a_n = \langle f, \phi_n \rangle_w / \langle \phi_n, \phi_n \rangle_w$, then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ converges to $f(x)$ if f is continuous at x, and to the mean value $\left[f(x^+) + f(x^-)\right]/2$ if f is discontinuous at x, for each point x in $a < x < b$.

Definition: (Periodic) S-L problem has the nonseparated (periodic) boundary conditions

$$y(a) = y(b), \qquad y'(a) = y'(b).$$

Definition: (Singular) S-L problem arises when $p(x)$ (and possibly $w(x)$) vanishes at one or both endpoints, so that $p(x) > 0$ and $w(x) > 0$ holds on open (a,b). Further, the boundary conditions are modified as follows:

(a) $p(a) = 0$ (and $p(b) \neq 0$): Then the boundary conditions are: y bounded at a, $\gamma y(b) + \delta y'(b) = 0$.

(b) $p(b) = 0$ (and $p(a) \neq 0$): Then the boundary conditions are: $\alpha y(a) + \beta y'(a) = 0$, y bounded at b.

(c) $p(a) = p(b) = 0$: Then the boundary conditions are: y bounded at a, y bounded at b.

Theorem: (Periodic and Singular S-L Theorem) Let λ_n and $\phi_n(x)$ denote any eigenvalue and corresponding eigenfunction of a periodic or a singular S-L problem.
(a) The eigenvalues are real,
(b) $q(x) \le 0$ on $[a,b]$ and $[p(x)\phi_n(x)\phi_n'(x)]_a^b \le 0$ for the eigenfunction $\phi_n(x)$, then not only is λ_n real, it is also nonnegative: $\lambda_n \ge 0$.
(c) Eigenfunction corresponding to distinct eigenvalues are orthogonal, that is, if $\lambda_j \ne \lambda_k$, then $\langle \phi_j, \phi_k \rangle_w = 0$.

Bessel Functions
They arise when solving PDEs in polar and cylindrical coordinates as eigenfunctions of the singular S-L problem (case (a))

$$(rR')' + (\lambda r - \tfrac{\nu^2}{r})R = 0, \quad 0 < r < a,$$

subject to the boundary conditions: $R = R(r)$ bounded as $r \to 0$ & $\gamma R'(a) + \delta R(a) = 0$.
The change of variables $x = \sqrt{\lambda} r$ and thus $d/dr = \sqrt{\lambda}\, d/dx$ yields

$$x^2 R'' + xR' + (x^2 - \nu^2)R = 0,$$

the Bessel's differential equation of order $\nu \ge 0$. For $x = 0$ being a RSP, we seek a Frobenius solution $y(x) = x^r \sum_{k=0}^{\infty} a_k x^k$. This results in the indicial equation $r^2 - \nu^2 = 0$ with roots $r = \pm\nu$.
The solution for root $r_1 = \nu$ is the Bessel function of the first kind of order ν:

$$J_\nu(x) = \left(\tfrac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!(k+\nu)!} \left(\tfrac{x}{2}\right)^{2k}$$

when ν is a positive integer. This can be generalized to

$$J_\nu(x) = \left(\tfrac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\tfrac{x}{2}\right)^{2k}$$

for $\nu \ge 0$. Here, $\Gamma(\nu)$ is the Gamma function defined by

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} dx .$$

as a generalization of the factorial operation to noninteger values. It can be shown by integration-by-parts that $\Gamma(\nu+1) = \nu\Gamma(\nu)$ and thus for ν a positive integer, $\Gamma(\nu+1) = \nu!$ while $\Gamma(1) = 1$.
The second independent solution for root $r_2 = -\nu$ requires separate considerations for ν **(1)** not an integer, **(2)** zero, **(3)** positive integer.
(1) When ν is not an integer, the second solution can be written as

$$J_{-\nu}(x) = \left(\tfrac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!\Gamma(k-\nu+1)} \left(\tfrac{x}{2}\right)^{2k}$$

because $\nu \to -\nu$ leaves the differential equation invariant. It is independent of $J_\nu(x)$, because $J_\nu(0) = 0$ while $\lim_{x\to 0^+} J_{-\nu}(x) = \infty$. Thus, a general solution is $R(x) = AJ_\nu(x) + BJ_{-\nu}(x)$.

Note also that the roots $r_1 - r_2 = \nu - (-\nu) = 2\nu$ differ by an integer when ν is one-half an odd integer $(1/2, 3/2, 5/2, \ldots)$. It can be shown, for example, that

$$J_{\frac{1}{2}}(x) = \sqrt{\tfrac{2}{\pi x}}\sin x \qquad \text{and} \qquad J_{-\frac{1}{2}}(x) = \sqrt{\tfrac{2}{\pi x}}\cos x .$$

These are called spherical Bessel functions.

(2) When ν is zero, the indicial roots are equal: $r_1 = r_2 = 0$. By Frobenius theorem, the second independent solution is sought in the form

$$R(x) = J_0(x)\ln(x) + \sum_{k=1}^{\infty} a_k x^k \quad \text{where} \quad J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} .$$

After some scaling, the Bessel Function of the second kind of order zero results

$$Y_0(x) = \frac{2}{\pi}\left\{ J_0(x)\left[\ln\left(\tfrac{x}{2}\right) + \gamma\right] + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k!)^2}\phi(k)\left(\frac{x}{2}\right)^{2k} \right\}$$

where, $\gamma = \lim_{k\to\infty}\left(\phi(k) - \ln k\right)$ is the Euler's constant and $\phi(k) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$.

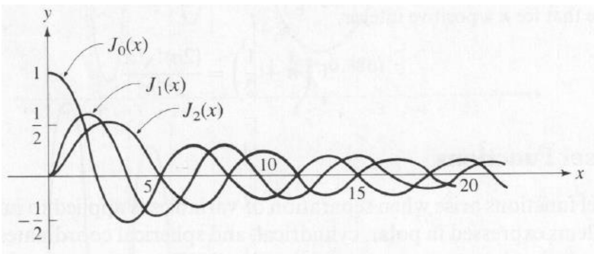
(3) When ν is a positive integer, the indicial roots differ by an integer. By Frobenius theorem, the second independent solution is sought in the form

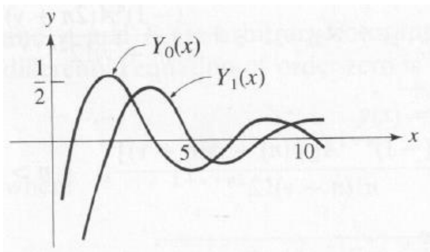
$$R(x) = \kappa J_\nu(x)\ln(x) + \sum_{k=0}^{\infty} a_k x^{k-\nu} .$$

After some scaling, the Bessel Function of the second kind of order ν results

$$Y_\nu(x) = \frac{2}{\pi}\left\{ J_\nu(x)\left[\ln\left(\tfrac{x}{2}\right) + \gamma\right] - \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\nu-1} \frac{(\nu-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k} - \frac{1}{2}\left(\frac{x}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k[\phi(k)+\phi(k+\nu)]}{k!(k+\nu)!}\left(\frac{x}{2}\right)^{2k} \right\} .$$

Graphs of some Bessel functions are shown below:





that can be constructed by the Matlab commands:

```
>> Jnu = besselj(nu,x);   % First kind Bessel fn. of order nu.
>> Ynu = bessely(nu,x);   % Second kind Bessel fn. of order nu.
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Due to the boundedness requirement as $r \to 0$, the general solution to the singular Bessel S-L problem is

$$R(r;\lambda) = AJ_\nu(\sqrt{\lambda}r), \quad 0 < r < a,$$

where the eigenvalue(s) are determined by the boundary conditions $\gamma R'(a;\lambda) + \delta R(a;\lambda) = 0$. The eigenfunctions $\{R(r;\lambda_n)\}_{n=1}^{\infty}$ form a basis set of orthogonal functions

$$\left(R(r;\lambda_n), R(r;\lambda_m)\right)_r \equiv \int_0^a R(r;\lambda_n)R(r;\lambda_m)\, r\, dr = \rho_n \delta_{nm} .$$

This allows Fourier-Bessel representation of piecewise-smooth functions $f(r)$ in terms of the Bessel S-L eigenfunctions as follows:

$$\tfrac{1}{2}\left(f(r^+)+f(r^-)\right)=\sum_{n=1}^{\infty}c_n R(r;\lambda_n) \quad \text{where} \quad c_n \equiv \tfrac{1}{\rho_n}\int_0^a f(r)\,R(r;\lambda_n)\,r\,dr\,.$$

The norm $\rho_n = \|R(r;\lambda_n)\|^2$ can be obtained from the Bessel DE as follows:

- Multiply the DE by $2rR'$ and rearrange: $0=\frac{d}{dr}(rR')^2+(\lambda r^2-\nu^2)\frac{d}{dr}(R)^2$
- Integrate to get: $\lambda\int_0^a r^2\frac{d}{dr}(R)^2dr=\left[\nu^2(R)^2-(rR')^2\right]_0^a$
- The norm follows: $\rho_n=\int_0^a R^2rdr=\frac{1}{2\lambda_n}\left[(rR')^2-\nu^2R^2+\lambda_n r^2R^2\right]_0^a$.

Recurrence Relations may be used to evaluate Bessel functions of higher orders by using the series representations. Some are:

- $J_{\nu-1}(x)+J_{\nu+1}(x)=\frac{2\nu}{x}J_\nu(x)$ for $\nu\ge 1$,
- $2J'_\nu(x)=J_{\nu-1}(x)-J_{\nu+1}(x)$ for $\nu\ge 1$,
- $J'_\nu(x)=-\frac{\nu}{x}J_\nu(x)+J_{\nu-1}(x)$ for $\nu\ge 1$,
- $J'_\nu(x)=\frac{\nu}{x}J_\nu(x)-J_{\nu+1}(x)$ for $\nu\ge 0$.

Further, multiplication of the last two equations by x^ν and $x^{-\nu}$ yields

- $\frac{d}{dx}\left(x^\nu J_\nu(x)\right)=x^\nu J_{\nu-1}(x)$ for $\nu\ge 1$,
- $\frac{d}{dx}\left(x^{-\nu} J_\nu(x)\right)=-x^{-\nu} J_{\nu+1}(x)$ for $\nu\ge 0$.

These results are also valid for $Y_\nu(x)$.

1. Solve for the eigenvalues and eigenfunctions, and work out the eigenfunction expansion of the given function f.
   (a) $y''+\lambda y=0,\ y(0)=0,\ y'(L)=0,\ f(x)=100$
   (b) $y''+\lambda y=0,\ y'(0)=0,\ y(L)=0,\ f(x)=1$
   (c) $y''+\lambda y=0,\ y'(0)=0,\ y'(L)=0,\ f(x)=\begin{cases}1, & 0\le x<L/2\\ 0, & L/2\le x\le L\end{cases}$

   How can you verify: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}$?

   (d) $y''+\lambda y=0,\ y'(0)=0,\ y(L)+y'(L)=0,\ f(x)=50$
   (e) $y''+\lambda y=0,\ y(0)+y'(0)=0,\ y(\pi)=0,\ f(x)=10$
   (f) $y''+\lambda y=0,\ y'(-1)=0,\ y'(1)=0,\ f(x)=\begin{cases}0, & -1\le x\le 0\\ 50, & 0<x\le 1\end{cases}$
   (g) $y''+\lambda y=0,\ y(0)-2y'(0)=0,\ y'(2)=0,\ f(x)=100$
   (h) $x^2y''+xy'+\lambda y=0,\ y(1)=0,\ y(a)=0,\ f(x)$

   Ans.: Let $\lambda_n=\alpha_n^2$. (a) $\{\alpha_n=(2n-1)\pi/2L;\ \sin(\alpha_n x)\}_{n=1}^{\infty}$, (b) $\{\alpha_n=(2n-1)\pi/2L;\ \cos(\alpha_n x)\}_{n=1}^{\infty}$, (c) $\{\alpha_n=n\pi/L;\ \cos(\alpha_n x)\}_{n=0}^{\infty}$,

(d) $\{\tan(\alpha_n L)=1/\alpha_n\,;\,\cos(\alpha_n x)\}_{n=1}^{\infty}$, (e) $\{\cot(\alpha_n\pi)=1/\alpha_n\,;\,\sin(\alpha_n(\pi-x))/\sin(\alpha_n\pi)\}_{n=1}^{\infty}$, (f) see (c),

(g) $\{\tan(2\alpha_n)=2/\alpha_n\,;\,\cos(\alpha_n(2-x))/\cos(2\alpha_n)\}_{n=1}^{\infty}$, (h) $\{\alpha_n=n\pi/\ln(a);\,\sin(\alpha_n\ln(x))\}_{n=1}^{\infty}$.

2. Recast each of the following differential equations in the S-L form.

   (a) $xy''+5y'+\lambda xy=0$, (b) $y''+2y'+xy+\lambda x^2y=0$, (c) $y''+y'+\lambda y=0$, (d) $y''-y'+\lambda xy=0$,

   (e) $x^2y''+xy'+\lambda x^2y=0$, (f) $y''+(\cot x)y'+\lambda y=0$

3. Find the adjoint L^* of the given operator L, i.e. $\langle L[u],v\rangle=\langle u,L^*[v]\rangle$, and state whether the given operator is self-adjoint ($L=L^*$) relative to the inner product chosen. If it is not, state why it is not. In each case, use the inner product $\langle f,g\rangle\equiv\int_0^1 f(x)g(x)dx$.

   (a) $L=\frac{d}{dx}$, $u(0)=0$ (b) $L=\frac{d}{dx}$, $u(1)=0$, (c) $L=\frac{d^2}{dx^2}$, $u(0)=u'(0)=0$,

   (d) $L=\frac{d^2}{dx^2}$, $u'(0)=u'(1)=0$, (e) $L=\frac{d^2}{dx^2}+3$, $u'(0)=u'(1)=0$, (f) $L=\frac{d^2}{dx^2}+\frac{d}{dx}$, $u(0)=u(1)=0$,

   (g) $L=\frac{d^3}{dx^3}-\frac{d^2}{dx^2}+2\frac{d}{dx}$, $u(0)=u'(0)=u'(1)=0$, (h) $L=\frac{d^2}{dx^2}-1$, $u(0)+u'(0)=u'(1)=0$,

   (i) $L=\frac{d^2}{dx^2}$, $u'(0)=u(1)+5u'(1)=0$, (j) $L=\frac{d^2}{dx^2}$, $u(0)-u(1)=u'(0)-u'(1)=0$,

   (k) $L=-\frac{d^2}{dx^2}$, $2u(0)-u(1)+4u'(1)=u(0)+2u'(1)=0$,

   (l) $L=-\frac{d^2}{dx^2}$, $u(0)-u(1)=u'(0)+u'(1)=0$

4. Show that the 4^{th} order Sturm-Liouville differential operator

$$\Im\equiv\frac{1}{w(x)}\left\{\frac{d^2}{dx^2}\left[s(x)\frac{d^2}{dx^2}\right]+\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x)\right\}$$

   is self-adjoint if the boundary conditions are homogeneous. State the boundary conditions.

5. Find the eigenvalues and eigenfunctions, and work out the eigenfunction expansion of the given function f.

   (a) $y''+\lambda y=0$, $y(0)=y(4)$, $y'(0)=y'(4)$, $f(x)=H(x-2)$, unit step function.

   (b) $y''+\lambda y=0$, $y(-1)=y(5)$, $y'(-1)=y'(5)$, $f(x)=x+2$,

   (c) $x^2y''+xy'+\lambda y=0$, $y(1)=y(2)$, $y'(1)=y'(2)$, $f(x)=6$,

   (d) $(1-x^2)y''-2xy'+\lambda y=0$, $y(0)=0$, $y(1)$ bounded, $f(x)=4$,

   (e) $(1-x^2)y''-2xy'+\lambda y=0$, $y'(0)=0$, $y(1)$ bounded, $f(x)=x$,

   (f) $(1-x^2)y''-2xy'+\lambda y=0$, $y(-1)$ bounded, $y'(0)=0$, $f(x)=5x^2$,

   (g) $(4-x^2)y''-2xy'+\lambda y=0$, $y(-2)$ bounded, $y(2)$ bounded, $f(x)=5-2x$

   Hint: $\{\lambda_n=n(n+1);\,P_n(x)\}_{n=0}^{\infty}$: (d) n:odd & $0\le x\le 1$, (e) n:even & $0\le x\le 1$, (f) n:even & $-1\le x\le 0$, (g) Transform to $[-1,1]$.

6. Put the power series solution about the ordinary point $x=0$, $y=\sum_{k=0}^{\infty}a_kx^k$ into the Legendre's equation $(1-x^2)y''-2xy'+\lambda y=0$ and derive the recursion formula $a_{k+2}=\frac{k(k+1)-\lambda}{(k+1)(k+2)}a_k$, $k=0,1,...$

Show that the series terminates for $\lambda = n(n+1)$, $n = 0,1,2,...$ and construct the first five Legendre polynomials, $P_n(x)$ scaled with $P_n(1) = 1$.

7. Use Rodrigues's formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$ to reproduce the first five Legendre polynomials, $P_n(x)$.
8. Expand the left-hand side of the Generating function: $(1-2xr+r^2)^{-1/2} = \sum_0^\infty P_n(x)r^n$ in a Taylor series in r, about $r=0$, up to r^3, to verify that the coefficients of $r^0,...,r^3$ are indeed $P_0(x),...,P_3(x)$.
9. Show by changing x to $-x$ in the Generating function that $P_n(-x) = (-1)^n P_n(x)$. What can you deduce about the values of $P_n(0)$ and $P_n'(0)$?
10. Show by taking $\partial/\partial r$ of the Generating function that one obtains the recursion formula $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$, $n = 2,3,...$
11. Show by taking $\partial/\partial x$ of the Generating function that one obtains the recursion formula $P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x) = P_{n-1}(x)$, $n = 2,3,...$
12. Recast the Legendre's equation $(1-x^2)y'' - 2xy' + \lambda y = 0$ in the S-L form and reason to verify the orthogonality relation $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$ for $n \neq m$.

    By squaring and integrating the Generating function from -1 to 1, and using the orthogonality to obtain

$$\int_{-1}^{1}(1-2xr+r^2)^{-1}dx = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}[P_n(x)]^2\,dx\right\}r^{2n},$$

    show that $\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1}$, $n = 0,1,2,...$
13. What can you say about the integral of the Legendre polynomials on the half interval, i.e. $\int_0^1 P_n(x)P_m(x)\,dx$?
14. Associated Legendre functions are soln's to the DE

$$((1-x^2)y')' + \left(n(n+1) - \frac{m^2}{1-x^2}\right)y = 0.$$

    where m is a nonnegative integer. It reduces to the Legendre DE when $m = 0$.

    Show that when $y(x)$ is a sol'n to the Legendre DE, $(1-x^2)^{m/2}\frac{d^m y}{dx^m}$ is a sol'n to the <u>Associated Legendre</u> DE. Thus, $P_{mn}(x) = (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}$ is called the <u>Associated Legendre functions</u> of degree n and order m. Show also that $P_{mn}(x)$ is nonvanishing only when $n \geq m$.
15. An alternative representation of associated Legendre functions is obtained by setting $x = \cos\theta$ for $0 \leq \theta \leq \pi$ where they take the form of trigonometric polynomials. Generate few of these functions using

$$P_{mn}(\cos\theta) = (-1)^m \sin^m\theta \frac{d^m P_n(\cos\theta)}{d(\cos\theta)^m},$$

and establish the orthogonality relations of $P_{mn}(\cos\theta)$ for m-fixed and n-fixed cases.

Hint: Re-write the associated Legendre DE in θ and identify the inner-product weight functions for each case.

16. Put the power series solution about the regular singular point $x=0$, $y=\sum_{k=0}^{\infty} a_k x^{k+r}$ into the Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ and derive the recursion relation

$$\sum_{k=0}^{\infty}\left\{\left[(k+r)^2 - \nu^2\right]a_k + a_{k-2}\right\}x^{k+r} = 0.$$

17. Show using the recursion relation for the case $\nu = 1/2$ that the Bessel functions of the first kind are obtained as $J_{1/2}(x) = \sqrt{2/(\pi x)}\sin x$ and $J_{-1/2}(x) = \sqrt{2/(\pi x)}\cos x$.

18. It can be shown that the Generating function for the Bessel function of the first kind, of order n, $J_n(x)$ is $\exp(\frac{x}{2}(t - \frac{1}{t})) = \sum_{n=-\infty}^{\infty} J_n(x)t^n$.

(a) Show by taking $\partial/\partial t$ of the Generating function that one obtains the recursion formula

$$J_{n+1}(x) = \frac{x}{2(n+1)}\left[J_n(x) + J_{n+2}(x)\right],\ n = 0,1,\ldots$$

(b) Show by taking $\partial/\partial x$ of the Generating function that one obtains the recursion formula

$$J'_n(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right],\ n = 1,2,\ldots$$

19. Show that equations in the form $\left[t^a u'\right]' + bt^c u = 0$ where a, b, c are real numbers, can be transformed to a Bessel equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ by transforming both independent and dependent variables by the change of variables $x = \alpha\sqrt{b}t^{1/\alpha}$ and $y = t^{-\nu/\alpha}u$ where $\alpha = 2/(c - a + 2)$ and $\nu = (1-a)/(c - a + 2)$. Apply this result to the following equations:

(a) $y'' + 4x^2y = 0$, (b) $xy'' - 2y' + xy = 0$, (c) $4y'' + 9xy = 0$, (d) $y'' + \sqrt[3]{x}y = 0$, (e) $4xy'' + y = 0$, (f) $4xy'' + 2y' + xy = 0$, (g) $y'' + xy = 0$, (h) $y'' + 4y = 0$ (compare with the known elementary solution).

20. Show that Bessel's equation of order zero has a solution J_0 which is analytic on the entire x-axis and satisfies the condition $J_0(0) = 1$.

21. Show that the Fourier-Bessel representation of the function $f(r) = a^2 - r^2$ in the form:

$$f(r) = \sum_{k=1}^{\infty} c_k J_0(\lambda_k r) \quad \text{where} \quad c_k \equiv \frac{1}{\rho_k}\int_0^a f(r) J_0(\lambda_k r)\, r\, dr$$

yields $c_k = \frac{4}{\lambda_k^2 J_0(\lambda_k a)}$ when the eigenvalue relation is $J_1(\lambda_k a) = 0$.

Hint: Use the relation $(x^{-\nu}J_\nu(x))' = -x^{-\nu}J_{\nu+1}(x)$, $\nu \ge 0$ and integration-by-parts.