EXERCISE SET #2

Main Theorem: (Frobenius) Let $x^2y'' + xq(x)y' + r(x)y = 0$ (#) be a 2nd order homogeneous linear differential equation whose coefficients are analytic in the interval $|x| < R$, let ν_1 and ν_2 be the roots of the indicial equation $\nu(\nu-1) + q(0)\nu + r(0) = 0$ and suppose that ν_1 and ν_2 have been labeled so that $\mathrm{Re}(\nu_1) \geq \mathrm{Re}(\nu_2)$. Then (#) has two linearly independent solutions y_1 and y_2, valid for $0 < |x| < R_0$, whose form depends upon ν_1 and ν_2 as follows:

Case 1. $\nu_1 - \nu_2$ not an integer. Then

$$y_1(x) = |x|^{\nu_1} \sum_{k=0}^{\infty} a_k x^k, \quad a_0 = 1,$$

$$y_2(x) = |x|^{\nu_2} \sum_{k=0}^{\infty} b_k x^k, \quad b_0 = 1.$$

Case 2. $\nu_1 = \nu_2 = \nu$. Then

$$y_1(x) = |x|^{\nu} \sum_{k=0}^{\infty} a_k x^k, \quad a_0 = 1,$$

$$y_2(x) = |x|^{\nu} \sum_{k=1}^{\infty} b_k x^k + y_1(x)\ln|x|.$$

Case 3. $\nu_1 - \nu_2$ a positive integer. Then

$$y_1(x) = |x|^{\nu_1} \sum_{k=0}^{\infty} a_k x^k, \quad a_0 = 1,$$

$$y_2(x) = |x|^{\nu_2} \sum_{k=0}^{\infty} b_k x^k + c y_1(x)\ln|x|, \quad b_0 = 1, \quad \text{c a (fixed) constant.}$$

The values of the constants in each of these solutions can be determined directly from the differential equation by substitution.

1. Determine the radius of convergence R of the given power series:

   (a) $\sum_{n=0}^{\infty} e^n x^n$, (b) $\sum_{n=0}^{1000} n!x^n$, (c) $\sum_{n=0}^{\infty} nx^n$, (d) $\sum_{n=0}^{\infty} (-1)^n n^{1000} x^n$, (e) $\sum_{n=0}^{\infty} (\frac{x+3}{2})^n$,

   (f) $\sum_{n=2}^{\infty} (n-1)^3 (x-5)^n$, (g) $\sum_{n=1}^{\infty} \frac{n^{50}}{n!}(x+7)^n$, (h) $\sum_{n=0}^{\infty} (n+1)x^{2n}$, (i) $\sum_{n=3}^{\infty} \frac{(-1)^n}{2^n}(x+2)^{2n}$,

   (j) $\sum_{n=100}^{\infty} (-1)^n \frac{n}{2^n}(x-5)^{2n}$

   Ans.: (a) $1/e$, (b) ∞, (c) 1, (d) 1, (e) 2, (f) 1, (g) ∞, (h) 1, (i) $\sqrt{2}$, (j) $\sqrt{2}$

2. Determine the radius of convergence R of the Taylor series of the given rational function $f(x)$ about each of the indicated points; you need not generate Taylor series.

   (a) $\frac{6}{x^2}, \quad x = -5, 1$, (b) $\frac{1}{x^2+x+1}, \quad x = -1, 0, 2$, (c) $\frac{x^4+x^3}{(x-1)(x-2)}, \quad x = 0, 50$, (d) $\frac{x^2+2x+3}{(x-1)(x^2+4)}, \quad x = -5, 5$,

   (e) $\frac{x^2+2x+3}{(x-1)^3(x^2+1)}, \quad x = -1, 0, 4$, (f) $\frac{6}{1+x+x^2+x^3}, \quad x = -3, 0, 3$, (g) $\frac{x^4+1}{(x-1)(x^2+5x+4)}, \quad x = -2, 0, 5$,

   (h) $\frac{x^3}{x^4+x^2-12}, \quad x = -4, 1, 8$, (i) $\frac{x^4+1}{x^2+1}, \quad x = 0, 2, 10$

   Ans.:(a) $x = 0; R = 5, 1$,(b) $x = \frac{1}{2}(-1 \mp i\sqrt{3}); R = 1, 1, \sqrt{7}$,(c) $x = 1, 2; R = 1, 48$,(d) $x = 1, \mp 2i; R = \sqrt{29}, 4$,(e) $x = 1, \mp i; R = \sqrt{2}, 1, 3$,(f) $x = -1, \mp i; R = 2, 1, 3$,

   (g) $x = -4, -1, 1; R = 1, 1, 4$, (h) $x = \mp\sqrt{3}, \mp 2i; R = 4 - \sqrt{3}, \sqrt{3} - 1, 8 - \sqrt{3}$, (i) $x = \mp i; R = 1, \sqrt{5}, \sqrt{101}$

3. Determine the least possible value of the radius of convergence of power series solutions about the specified point x_0; you need not construct the series.

   (a) $y'' - 3y' + 6y = 0, \quad x_0 = 6$, (b) $y'' - e^x y = 0, \quad x_0 = 0$, (c) $xy'' + y' + x^2 y = 0, \quad x_0 = 5$,

   (d) $(x^2 - 1)y'' - (x+1)y' + (x-1)y = 0, \quad x_0 = 0.3$, (e) $(4x^2 + 1)y'' + 3xy' + y = 0, \quad x_0 = 5$,

   (f) $(x^2 + 2x + 5)y'' + (x^2 + 1)y = 0, \quad x_0 = 1$, (g) $(1 - x^2)y'' - 2xy' + 6y = 0, \quad x_0 = 0$,

   (h) $xy'' + y' + y = 0, \quad x_0 = 4$, (i) $(x^2 + 4x)y'' + y' - y = 0, \quad x_0 = -3$,

   (j) $(x^3 - 8)y'' - 2y = 0, \quad x_0 = 5$

   Ans.: (a) ∞, (b) ∞, (c) 5, (d) 0.7, (e) $\sqrt{101}/2$, (f) $2\sqrt{2}$, (g) 1, (h) 4, (i) 1, (j) 3

4. Express the general solution of each of the following equations as a power series about the point $x = 0$ and specify an interval in which the solution is valid.

   (a) $y'' - 3xy = 0$, (b) $y'' - x^2 y = 6x$, (c) $3y'' - xy' + y = x^2 + 2x + 1$, (d) $(x^2 - 1)y'' + xy' - 4y = 0$, (e) $(x^2 - 2)y'' + xy' - y = x^2$, (f) $y'' + \frac{x}{x^2-4} y' - \frac{25}{x^2-4} y = \frac{1+2x}{x^2-4}$,

   Ans.: (a) $y_1 = 1 + \sum_{k=1}^{\infty} \frac{1}{k![2 \cdot 5 \cdot 8 \cdots (3k-1)]} x^{3k}$; $y_2 = \sum_{k=0}^{\infty} \frac{1}{k![1 \cdot 4 \cdot 7 \cdots (3k+1)]} x^{3k+1}$; ∞,

   (c) $y_1 = 1 - \sum_{k=1}^{\infty} \frac{1}{k!(2k-1)} (\frac{1}{6})^k x^{2k}$; $y_2 = x$; $y_p = \frac{1}{6} x^2 + \sum_{k=2}^{\infty} \frac{7}{k!(2k-1)} (\frac{1}{6})^k x^{2k} + \sum_{k=1}^{\infty} \frac{(k-1)!}{(2k+1)!} (\frac{2}{3})^k x^{2k+1}$; ∞,

   (d) $y_1 = 1 + \sum_{k=1}^{\infty} \frac{[(0)(-1)-1][(1)(0)-1][(2)(1)-1]\ldots[(k-1)(k-2)-1]}{(2k)!} (2x)^{2k}$; $y_2 = x + \sum_{k=1}^{\infty} \frac{[(1)(-1)-4][(3)(1)-4][(5)(3)-4]\ldots[(2k-1)(2k-3)-4]}{(2k+1)!} (x)^{2k+1}$; (-1,1),

   (f) $y_1 = x - x^3 + \frac{1}{5} x^5$; $y_2 = 1 - \frac{25}{8} x^2 + \frac{175}{128} x^4 - \frac{105}{1024} x^6 - 5 \sum_{k=4}^{\infty} \frac{(2k-7)!(2k+1)(2k+3)}{2^{4(k-1)} k!(k-4)!} x^{2k}$; $y_p = -\frac{1}{8} x^2 - \frac{1}{12} x^3 + \frac{7}{128} x^4 + \frac{1}{60} x^5 - \frac{21}{5120} x^6 - \frac{1}{5} \sum_{k=4}^{\infty} \frac{(2k-7)!(2k+1)(2k+3)}{2^{4(k-1)} k!(k-4)!} x^{2k}$; (-2,2)

5. Find the first four nonzero terms in the series expansion of the solution of each of the following IVP, and determine a (minimal) interval of convergence for the series.

   (a) $y'' + xy' - 2y = e^x$, $y(0) = y'(0) = 0$.

   (b) $xy'' + y' + xy = 0$, $y(1) = 0$, $y'(1) = -1$.

   (c) $3xy'' - y' = 0$, $y(-2) = 1$, $y'(-2) = -1$.

   (d) $(\cos x)y'' + 2xe^x y = 0$, $y(0) = 0$, $y'(0) = 1$.

   Hint: You may use Cauchy product of two series $\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{k=0}^{\infty} c_k x^k$ where $c_k = \sum_{j=0}^{k} a_j b_{k-j}$ where necessary.

   Ans.: (a) $y = \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \sum_{k=3}^{\infty} a_{2k} x^{2k} + \sum_{k=2}^{\infty} a_{2k+1} x^{2k+1}$; ∞,

6. Consider the Airy's equation: $y'' + xy = 0$.

   (a) Derive a general solution in power series about $x_0 = 0$ up to the terms by which the pattern of the coefficients should become clear. What are the radii of convergence of your series solutions?

   (b) Alternatively, use MATLAB ODE solver `ode45` to obtain graphs of two linearly independent solutions. Let the first satisfy the initial conditions $y(0) = 1$, $y'(0) = 0$, and the second satisfy the initial conditions $y(0) = 0$, $y'(0) = 1$. Plot these solutions on $-5 \le x \le 20$, $-5 \le y \le 5$.

   (c) Discuss the qualitative nature of the graphs obtained in (b) in terms of the form of the differential equation and explain why Airy's equation is called a change-of-type differential equation.

Hint: Think of the nature of the solutions of the simpler constant-coefficient equation $y''+cy=0$, for negative and positive values of the constant c.

Ans.: (a) $y_1=1+\sum_{k=1}^{\infty}\frac{1}{2\cdot3\cdots(3k-1)(3k)}x^{3k}$; $y_2=x+\sum_{k=1}^{\infty}\frac{1}{3\cdot4\cdots(3k)(3k+1)}x^{3k+1}$; ∞ ,

7. Find a necessary and sufficient condition that a differential equation of the form $(x^2+\alpha)y''+\beta xy'+\gamma y=0$ has a polynomial solution of degree n.

Ans.: There exists a solution which is a polynomial of degree, (i) $n=0$ iff $\gamma=0$, (ii) $n=1$ iff $\beta+\gamma=0$, (iii) $n\geq 2$ iff $n(n-1)+\beta n+\gamma=0$. In fact, for an equation of hypergeometric type, i.e. $p(x)y''+q(x)y'+\lambda y=0$, where $p(x)$ and $q(x)$ are at most 2nd and 1st order polynomials, respectively, and λ is a constant, when $\lambda+nq'+\frac{1}{2}n(n-1)p''=0$, the equation of hypergeometric type has a particular solution which is a polynomial of degree n.

8. Find the first four nonzero terms in the power series expansion of $\int_0^x \exp(-t^2)dt$. What is the interval of convergence of the power series expansion of this integral?

Ans.: $x-\frac{1}{3}x^3+\frac{1}{10}x^5-\frac{1}{42}x^7+\ldots$; ∞ .

9. Let $y(x)=\sum_{k=0}^{\infty}a_k x^k$ be a solution of the equation $y''+p(x)y'+q(x)y=0$ in the interval $|x|<r$, and suppose that $p(x)=\sum_{k=0}^{\infty}p_k x^k$ and $q(x)=\sum_{k=0}^{\infty}q_k x^k$ in this interval. Show that

$$a_{k+2}=-\frac{1}{(k+1)(k+2)}\sum_{j=0}^{k}\left[(j+1)p_{k-j}a_{j+1}+q_{k-j}a_j\right] \text{ for } k=0,1,2,\ldots$$

10. Find and classify all of the singular points for the following equations:

(a) $x^3(x^2-1)y''-x(x+1)y'-(x-1)y=0$, (b) $(x^4-1)y''+xy'=0$, (c) $(3x-2)^2xy''+xy'-y=0$, (d) $(x+1)^4(x-1)^2y''-(x+1)^3(x-1)y'+y=0$, (e) $x^3(x-1)y''+(x-1)y'+2xy=0$,

(f) $x^3(x+1)^2y''-y=0$, (g) $x(1-x)y''+(1-5x)y'-4y=0$.

Ans.: (a) $x=\pm1$ RSP; $x=0$ ISP, (b) $x=\pm1$ RSP, (e) $x=1$ RSP; $x=0$ ISP, (g) $x=0,1$ RSP.

11. Show that the equation $x^3y''+y=0$ has no nontrivial series solution of the form $\sum_{k=0}^{\infty}a_k x^{k+\nu}$ for any real number ν.

12. Find the indicial equation associated with the regular singular point at $x=0$ for each of the following equations:

(a) $x^2y''+xy'-y=0$, (b) $x^2y''-2x(x+1)y'+(x-1)y=0$, (c) $x^2y''-2xy'+y=0$,

(d) $x^2y''-xy'+(x^2-\lambda^2)y=0$, λ a constant, (e) $xy''+(1-x)y'+\lambda y=0$, λ a constant.

Ans.: (a) $\nu^2-1=0$, (b) $\nu^2-3\nu-1=0$, (c) $\nu^2-3\nu+1=0$, (d) $\nu^2-2\nu-\lambda^2=0$, (e) $\nu^2=0$.

13. Show that the indicial equation associated with $x^2y''+xq(x)y'+r(x)y=0$ is $\nu(\nu-1)+q(0)\nu+r(0)=0$ whenever q and r are polynomials.

14. (a) Find the indicial equation associated with each of the regular singular points $x=1$ and $x=-1$ for Legendre's equation $(1-x^2)y''-2xy'+\lambda y=0$.

(b) Use power series approach about the regular point $x=0$, to show that the Legendre's equation has polynomial solutions for certain λ.

Ans.: (a) $\nu^2 = 0$ in each case (b) $\lambda = n(n+1)$, Legendre polynomials $P_n(x)$: $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2-1)$, $P_3 = \frac{1}{2}(5x^3-3x)$, …

15. Find two linearly independent solutions of the equations that are convergent whenever $|x| > 0$:

(a) $2x^2y'' + xy' - y = 0$, (b) $9x^2y'' + 3x(x+3)y' - (4x+1)y = 0$, (c) $xy'' + \frac{1}{2}(x+1)y' - y = 0$,

(d) $8x^2y'' - 2x(x-1)y' + (x+1)y = 0$, (e) $4x^2y'' + x(2x-7)y' + 6y = 0$.

Ans.: (b) $y_1 = |x|^{1/3}(1+\frac{1}{5}x)$; $y_2 = |x|^{-1/3}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!3^k(3k-5)(3k-2)}x^k$, (d) $y_1 = |x|^{1/2}$; $y_2 = |x|^{1/4}\sum_{k=0}^{\infty}\frac{1}{k!2^{2k}(4k-1)}x^k$.

16. Show that Laguerre's equation of order ν, $xy'' + (1-x)y' + \nu y = 0$, has a solution which is analytic for all x, and which reduces to a polynomial when ν is a non-negative integer.

Ans.: $y = a_0\left[1+\sum_{k=1}^{\infty}\frac{\nu(\nu-1)\cdots(\nu-k+1)}{(k!)^2}(-x)^k\right]$. If $\nu = 0$, $y = a_0$; if ν is a positive integer, then $y = a_0\sum_{k=0}^{\nu}\frac{\nu!}{(\nu-k)!(k!)^2}(-x)^k$.

17. **Compute the values of the coefficients a_1, a_2, a_3 in the series solutions of each of the following equations: (assume $a_0 = 1$)

(a) $x^2y'' + x(x+1)y' + y = 0$, (b) $16x^2y'' - 4x(x^2-4)y' - y = 0$, (c) $x^2(x^2-1)y'' - xy' - 2y = 0$,

(d) $8x^2(x-2)y'' + 2xy' - (\cos x)y = 0$, (e) $x^2y'' + xe^xy' + y = 0$.

Ans.: (a) $\nu = \pm i$; $a_1 = -\frac{2}{5} \mp \frac{1}{5}i$; $a_2 = \frac{1}{10} \pm \frac{1}{20}i$; $a_3 = -\frac{17}{780} \mp \frac{1}{130}i$, (c) $\nu = \pm\sqrt{2}i$; $a_1 = 0$; $a_2 = \frac{1}{12}(-4 \pm \sqrt{2}i)$; $a_3 = 0$, (e) $\nu = \pm i$; $a_1 = -\frac{2}{5} \mp \frac{1}{5}i$; $a_2 = \frac{1}{80}(3 \mp i)$; $a_3 = \frac{1}{9360}(67 \pm 81i)$. Note that $x^{\alpha+i\beta} = x^{\alpha}[\cos(\ln(\beta x)) + i\sin(\ln(\beta x))]$ where real and imaginary parts form linearly independent solutions.

18. **Find two linearly independent solutions on the positive axis for each of the following equations:

(a) $x^2y'' + x(x-1)y' + (1-x)y = 0$, (b) $xy'' + (1-x)y' - y = 0$, (c) $x^2y'' + 3xy' + (x+1)y = 0$,

(d) $x^2y'' + 2x^2y' - 2y = 0$, (e) $xy'' - (x+3)y' + 2y = 0$, (f) $xy'' + (2x+3)y' + 4y = 0$,

(g) $xy'' + (x^3-1)y' + x^2y = 0$, (h) $x^2y'' - 2x^2y' + 2(2x-1)y = 0$, (i) $xy'' + (1-x)y' + 3y = 0$,

(j) $x^2y'' + x^2y' + (3x-2)y = 0$.

Ans.: (a) $y_1 = x$; $y_2 = x\ln x + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!k}x^{k+1}$, (c) $y_1 = x^{-1}\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}x^k\right]$; $y_2 = x^{-1}\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}x^k\right]\ln x + 2x^{-1}\left[\sum_{k=1}^{\infty}\frac{(1+2^{-1}+\cdots+k^{-1})(-1)^{k+1}}{(k!)^2}x^k\right]$,

(e) $y_1 = 4!\sum_{k=0}^{\infty}\frac{k+1}{(k+4)!}x^{k+4}$; $y_2 = 1+\frac{2}{3}x+\frac{1}{6}x^2$, (g) $y_1 = x^2 + \sum_{k=1}^{\infty}\frac{(-1)^k}{5\cdot 8\cdots(3k+2)}x^{3k+2}$; $y_2 = \sum_{k=0}^{\infty}\frac{(-1)^k}{3^k k!}x^{3k}$, (i) $y_1 = 1-3x+\frac{3}{2}x^2-\frac{1}{6}x^3$;

$y_2 = 7x - \frac{23}{4}x^2 + \frac{11}{12}x^3 - 6\sum_{k=4}^{\infty}\frac{(k-4)!}{(k!)^2}x^k + (1-3x+\frac{3}{2}x^2-\frac{1}{6}x^3)\ln x$.

19. Consider the differential equation $y'' + \sqrt{x}y = 0, \quad 0 < x < \infty$.

(a) Show that it has an irregular singular point at $x = 0$, so the Frobenius theory does not apply.

(b) Show that the change of variables $\sqrt{x} = t$ results in the differential equation $tY'' - Y' + 4t^4Y = 0$ where $y(x(t)) \equiv Y(t)$ and that it has a regular singular point at $t = 0$.

(c) Construct Frobenius-type solutions of the differential equation in (b) and then replace variable t in that solution by $\sqrt{x}$ to obtain the solution

$$y(x) = c_1(1 - \tfrac{4}{15}x^{5/2} + \tfrac{1}{75}x^{10/2} - \ldots) + c_2(1 - \tfrac{4}{35}x^{5/2} + \tfrac{2}{525}x^{10/2} - \ldots).$$

Are the series solutions above of the Frobenius form, $y(x) = x^{\nu}\sum_{k=0}^{\infty}a_kx^k$? Explain.

20. **Consider single-lane driving with no passing, and suppose the velocities of our car, and the car ahead of us, are $v(t)$ and $v_a(t)$, respectively. As a simple model, suppose that we accelerate and decelerate proportional to the perceived velocity difference, $\frac{dv}{dt} = k[v_a(t) - v(t)]$ in which k is a constant that could be determined empirically and assumed known. Suppose now that the car ahead stops abruptly so $v_a(t) = 0$ for all $t \geq 0$ and the question is whether the car behind (our car) can stop in time to avoid a collision. Since $v_a(t) = 0$ for all , the differential model becomes $\frac{dv}{dt} = -kv(t)$ which is easy to solve. To be more realistic, let us build into the model a nonzero driver reaction time, say δ, so that the model becomes $\frac{dv}{dt} = -ku_\delta(t)v(t-\delta)$ where $u_\delta(t)$ is the unit step function (Is this differential equation linear?). This is called a delay-type differential equation with delayed response time δ.

(a) Show that application of the Laplace transform to the delay-type model gives $V(s) = v_0 \big/ (s + ke^{-\delta s})$ for the initial condition $v(0) = v_0$ where $V(s) = L\{v(t)\}$.

(b) In order to invert (otherwise difficult), re-express $V(s) = \left(\frac{v_0}{s}\right)\left(1 \big/ \left(1 + k\left(e^{-\delta s}/s\right)\right)\right)$ (Why?) and use the binomial series $\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \ldots$ (valid for $|z| < 1$) to obtain

$$V(s) = v_0\left[\frac{1}{s} - k\frac{e^{-\delta s}}{s^2} + k^2\frac{e^{-2\delta s}}{s^3} - \ldots\right]$$

and, in turn, after inversion to obtain the series solution

$$v(t) = v_0\left[1 - ku_\delta(t)(t-\delta) + \frac{1}{2!}k^2u_{2\delta}(t)(t-2\delta)^2 - \frac{1}{3!}k^3u_{3\delta}(t)(t-3\delta)^3 + \ldots\right]$$

(c) In order to find the stopping distance, replace $v(t) = \frac{dx}{dt}$ in the solution above and integrate from 0 to t with $x(0) = 0$ to show that

$$x(t) = v_0\left[t - \frac{1}{2!}ku_\delta(t)(t-\delta)^2 + \frac{1}{3!}k^2u_{2\delta}(t)(t-2\delta)^3 - \ldots\right].$$

(c) For the case of zero reaction time $\delta = 0$, show that the series above can be summed to obtain

$$x(t) = \frac{v_0}{k}(1 - e^{-kt}).$$

(d) As a numerical experiment, let $k = 1$ and $v_0 = 50$, and consider the reaction times $\delta = 0, 0.3, 0.6, 1$ seconds. Plot $x(t)$ versus t, to determine the safe distance to avoid hitting the car ahead for each of those values of δ.

**Challenging Problems