EXERCISE SET #1

1. (a) Show that e^{ax} and xe^{ax} are solutions of the differential equation $(D-a)^2 y = 0$ where D denotes differentiation operator, i.e. $Dy = y'$, $D^2 y = y''$. Is the sum of these functions a solution? Justify.

   (b) Show that $\frac{1}{9}x^3$ and $\frac{1}{9}(x^{3/2}+1)^2$ are solutions of the differential equation $(Dy)^2 - xy = 0$. Is the sum of these functions a solution? Justify.

2. Given the nonhomogeneous differential equation: $\left[(1-x^2)D^2 - 2xD\right]y = 2x, \quad -1 < x < 1$

   (a) Show that $y_h = c_1 + c_2 \ln\left(\frac{1+x}{1-x}\right)$ is the general solution of the associated homogeneous equation.

   (b) Construct Green's function for the given linear differential operator: $(1-x^2)D^2 - 2xD$.

   (c) Find a particular solution using the Green's function.

3. Consider the second-order normal homogeneous linear differential equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

   (a) Identify p and q so that it can be written in the self-adjoint form

$$\left(p(x)y'\right)' + q(x)y = 0$$

   (b) Show that if y_1 and y_2 are solutions to the differential equation, then $p\left[y_1y_2' - y_1'y_2\right]$ is a constant.

   (c) Use (b) to construct the formula to determine the second solution:

$$y_2(x) = y_1(x)\int \frac{e^{-\int [a_1(x)/a_2(x)]\,dx}}{\left(y_1(x)\right)^2}\,dx$$

   (d) Verify that $(1-x^2)y'' - 2xy' + 2y = 0$ has $y = x$ as a solution. Find the second solution in the interval $-1 < x < 1$ in which the equation is normal.

4. The functions $f(x)$ and $g(x)$ are linearly independent on some interval I, whenever their Wronskian $W\left[f(x), g(x)\right] = fg' - f'g$ is not identically zero on I. Show that the Wronskian of $f(x) = x^3$ and $g(x) = |x|^3$ is identically zero on $-\infty < x < \infty$, yet they are clearly linearly independent (not constant multiple of each other) in $-\infty < x < \infty$. Why?

5. Obtain a general solution. The x interval is $-\infty < x < \infty$ unless specified otherwise.

   (a) $y'' - 2y' + y = 6x^2$, (b) $y'' + y = \sec x$, $-\pi/2 < x < \pi/2$, (c) $y'' - 2y' = 3\cosh x$,

(d) $y'' + 4y' + 4y = e^{-2x}/x^2$, $x > 0$, (e) $x^2y'' - 2y = 3x^2$, $x > 0$, (f) $xy'' - 4y' = 36x^3 - 30x$, $x > 0$, (g) $(\cos x)y'' + (\sin x)y' = \sin x$, $y_1(x) = 1$, $y_2(x) = \sin x$, $-\pi/2 < x < \pi/2$,

(h) $(x-1)y'' - xy' + y = 2x - 2 - x^2$, $y_1(x) = x$, $y_2(x) = e^x$, $x > 1$, (i) $xy'' - (x+1)y' + y = 0$, $y_1(x) = 1 + x$, $x > 0$, (j) $(1-x^2)y'' - 2xy' + 2y = 0$, $y_1(x) = x$, $-1 < x < 1$,

(k) $y'' + 4y' + 3y = 60\sin 3x$, (l) $x^2y'' - 2xy' - 10y = 20x^3$, $x > 0$.

6. Show that $L\left\{\int_0^t f(u)\,du\right\} = \dfrac{F(s)}{s}$ where $L\{f(t)\} = F(s)$ is the Laplace Transform of $f(t)$.

7. Show that $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$ provided that $\lim_{t\to 0} \dfrac{f(t)}{t}$ exists where $L\{f(t)\} = F(s)$.

8. Show that $L\{g(t)\} = e^{-as}F(s)$ for $g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$ where $L\{f(t)\} = F(s)$.

9. Show that $L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$ where $L\{f(t)\} = F(s)$.

10. Show that $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$ for $n = 1, 2, 3, \ldots$ where $L\{f(t)\} = F(s)$.

11. If f is periodic with period T on $0 \le t < \infty$ and piecewise continuous on one period,

(a) Show that $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt$ for $s > 0$.

(b) Apply Laplace transform to solve the IVP: $y' + y = f(t)$, $y(0) = 0$ for periodic $f(t)$ defined over one period as 1 on $0 < t < 2$, 0 on $2 < t < 3$.

12. Use Laplace transform to solve the equation

$$y' + 2y + \int_0^t y(\tau)\,d\tau = \begin{cases} t, & t < 1 \\ 2 - t, & 1 \le t \le 2 \\ 0, & t > 2 \end{cases}$$

subject to the initial condition $y(0) = 1$.

13. Show that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$ by Laplace Transform. Hint: Consider $L\{\frac{\sin t}{t}\}\big|_{s\to 0}$.

14. The Gamma function is defined by $\Gamma(p)=\int_0^\infty u^{p-1}e^{-u}du$ for $p>0$. Use integration by parts to show that $\Gamma(p+1)=p\Gamma(p)$ for all $p>0$ and in particular $\Gamma(n+1)=n!$ for positive integer n.

15. Show that $L\left\{t^p\right\}=\frac{1}{s^{p+1}}\Gamma(p+1)$ for all $p>-1$ and $s>0$.

16. **Suppose one takes a dose of a certain drug, either orally or intravenously. As the blood circulates, the concentration $c(t)$ of the drug will tend to become uniform throughout the circulatory system. That will probably happen so quickly, compared to the time T between doses, that we can idealize the situation and model the drug inputs as delta functions. Studies show that following a dose the concentration diminishes with time approximately according to the equation $dc/dt=-kc$, where k is a positive experimentally known constant. Thus, the complete problem can be modelled as follows:

$$c'+kc=f(t),\quad c(0)=0$$

where $f(t)=C_1\left[\delta(t)+\delta(t-T)+\delta(t-2T)+...\right]$ and where C_1 is the increase in concentration due to one dose.

(a) Derive the solution $c(t)=C_1\left[e^{-kt}+e^{-k(t-T)}u(t-T)+e^{-k(t-2T)}u(t-2T)+...\right]$.

(b) Generate a computer plot of $c(t)$ on $0<t<3.5$ using $k=T=C_1=1$.

Hint: In Matlab, type

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t = linspace(0,3.5,30);
c = exp(-t)+exp(-(t-1)).*(t>=1)+exp(-(t-2)).*(t>=2)+exp(-(t-3)).*(t>=3);
plot(t,c)
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(c) Show that $c(t)$ approaches to a periodic function as $t\to\infty$ with period T.

Hint: Show that $c(t)=C_1e^{-kt}s_n$ for $nT\le t<(n+1)T$ where $s_n=1+e^{kT}+e^{2kT}+...+e^{nkT}=(1-e^{(n+1)kT})/(1-e^{kT})$ and then show that $\Delta c=\left[c(t+T)-c(t)\right]\downarrow 0$ as $t\uparrow\infty$.

(d) Show that as $t\to\infty$ the maxima and minima approach to these values:

$$c_{max}=C_1\frac{1}{1-e^{-kT}},\qquad c_{min}=C_1\frac{e^{-kT}}{1-e^{-kT}}.$$

Hint: Show that as $t\uparrow\infty$: $c(t)\uparrow c_{max}$ as $t\downarrow nT$ and $c(t)\downarrow c_{min}$ as $t\uparrow(n+1)T$.

17. **Consider an harmonic oscillator forced periodically by hammer blows with period T. Specifically, consider the initial value problem (IVP)

$$x''+x=\delta(t)+\delta(t-T)+\delta(t-2T)+...,\quad \text{with}\quad x(0)=x'(0)=0.$$

Construct the solution and show whether or not the response $x(t)$ will be resonant if $T=2\pi$, that is, if the hammer blows occur at the natural frequency.

Hint: Show that $x(t)=\sin(t)+\sin(t-T)u(t-T)+\sin(t-2T)u(t-2T)+...$ and set $T=2\pi$.

18. Find the impulse response function and construct the Green's function for

$$x'' - 2x' + 2x = f(t).$$

Use the Green's function to obtain the solution for $f(t) = \sin t$. Do you observe resonance?

19. **Consider a flexible string stretched by a tension τ, tied at its two ends, and subjected to a point force F at $x = A$. Neglecting the weight of the string (compared to F), and restricting F to be small enough so that the deflection at $x = A$ is small compared to L, it can be shown that the deflection $y(x)$ is governed by the boundary value problem (BVP)

$$\tau y'' = F\delta(x - A), \quad \text{with} \quad y(0) = y(L) = 0.$$

Construct the solution by:

(a) Considering the problem in two separate domains, namely, (i) $0 < t < A$ and (ii) $A < t < L$, and then matching the two solutions at $x = A$ (Note the jump discontinuity $\tau y' \big|_{A^-}^{A^+} = F$ at $x = A$).

Hint: Solve $\tau y'' = 0$, $y(0) = 0$, $y(A) = y_0$ in $0 < t < A$ and $\tau y'' = 0$, $y(A) = y_0$, $y(L) = 0$ in $A < t < L$ and match.

(b) Using Laplace transform. Note that this is a BVP, so how can one use Laplace transform for a BVP?

Hint: Set $y'(0) = c$ where unknown constant c may, somehow, be determined by the boundary condition at $x = L$.

20. Note that Laplace transform is particularly convenient if the initial conditions are specified at $t = 0$. How can you apply Laplace transform if the conditions are not specified at $t = 0$? Try on the problem: $y'' + 4y = 8t, \quad y(\pi) = y'(\pi) = 0$.

Hint: Set $y(0) = a$ and $y'(0) = b$ where unknown constants a and b are to be determined as suggested above.

21. What about the finiteness of the interval in a BVP, such as, $y = 3 + x + x^2$ on $0 < x < 2$ with the boundary conditions $y(0) = 1$ and $y(2) = 7$. Besides compensating for the missing initial condition at $x = 0$ as suggested above, one can extend the interval $0 < x < \infty$. This automatically implies extending the forcing function to the whole interval $0 < x < \infty$. Justify that the extended definition of the forcing function is immaterial on the BVP, say, $y'' + y = 3 + x + x^2$ with the boundary conditions $y(0) = 1$ and $y(2) = 7$.

22. Consider the motion of coupled two-mass mechanical oscillator modelled by

$$y_1'' + 2y_1 - y_2 = 50, \quad y_1(0) = y_1'(0) = 0,$$
$$y_2'' - y_1 + 2y_2 = 50, \quad y_2(0) = y_2'(0) = 0.$$

Solve for $y_1(t)$ and $y_2(t)$ using the Laplace transform.

23. ******The integral $T[f(t)] = \int_\alpha^\beta f(t)K(t,s)\,dt \equiv F(s)$ is called an integral transform; the input is $f(t)$ and the output is its transform $F(s)$. For the Laplace transform the kernel $K(t,s)$ is e^{-st}, the limits are $\alpha = 0$ and $\beta = \infty$ so that the generic transform T becomes $T[f(t)] = L[f(t)]$.

The key property of the Laplace transform is that the transform of the derivative $f'(t)$ be linear in $F(s)$, i.e. $T[f'(t)] = a(s)F(s) + b(s)$, so that it converts linear constant-coefficient differential equations to linear algebraic equations.

(a) Show that the above form of $T[f'(t)]$ leads to the choice $K(t,s) = e^{-st}$ and hence to the Laplace transform.

(b) Now design a "Cauchy-Euler" transform for Cauchy-Euler equations, $at^2x'' + btx' + cx = g(t)$ on the interval $1 < t < \infty$, i.e. choose the kernel $K(t,s)$ in $C[f(t)] = \int_1^\infty f(t)K(t,s)\,dt$ so that $C[tf'(t)] = a(s)F(s) + b(s)$. Show that the suitable kernel is $K(t,s) = t^{-s}$.

(c) Drive necessary "Cauchy-Euler" transform formulas, and apply them to solve

$$t^2x'' - 2tx' + 2x = 2t^3, \quad x(1) = x'(1) = 0.$$

******Challenging Problems