SUPPLEMENTARY PROBLEMS 0

1. We can obtain qualitative information about the solutions to first order differential equation by drawing its direction (flow) field. Use the qualitative approach with the following equations and plot the given solution over its direction field:

   (a) $y' = e^y, \quad y(0)=1, \; 0 \le t \le 0.20$, whose exact solution is $y(t) = \ln(e/(1-et))$.

   (b) $y' = \dfrac{1}{t^2} - \dfrac{y}{t} - y^2, \quad y(1) = -1, \; 1 \le t \le 2$, whose exact solution is $y(t) = -1/t$.

   (c) $y' = \dfrac{2}{t} y + t^2 e^t, \quad y(1) = 0, \; 1 \le t \le 2$, whose exact solution is $y(t) = t^2(e^t - e)$.

   (d) $u' = 3u + 2v, \quad u(0) = 0; \quad v' = 4u + v, \quad v(0) = 1, \quad 0 \le t \le 1$, whose exact solution is $u(t) = \frac{1}{3}(e^{5t} - e^{-t})$, $v(t) = \frac{1}{3}(e^{5t} + 2e^{-t})$.

   (e) $u' = -4u - 2v + \cos t + 4\sin t, \quad u(0) = 0; \quad v' = 3u + v - 3\sin t, \quad v(0) = -1, \quad 0 \le t \le 2$, whose exact solution is $u(t) = 2e^{-t} - 2e^{-2t} + \sin t$, $v(t) = -3e^{-t} + 2e^{-2t}$.

   Note : The following sequence of Matlab commands draws a direction field and the solution for the system:

(a)
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[T, Y] = meshgrid(0:0.02:0.20, 1:0.02:1.8);
FY = exp(Y); % direction field
L = sqrt(1 + FY.^2); % arrow length
t = linspace(0,0.2,100);
YE = log(exp(1)./(1-exp(1)*t));
quiver(T, Y, 1./L, FY./L, 0.5)
axis equal tight, hold on
plot(t,YE)
xlabel 't'
ylabel 'y'
title 'Vector field and the solution for the equation (a)'
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(d)
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[U, V] = meshgrid(0:5:50, 0:5:50);
FU = 3*U + 2*V;
FV = 4*U + V;
t = linspace(0,1,100);
UE = (1/3)*(exp(5*t)-exp(-t));
VE = (1/3)*(exp(5*t)+2*exp(-t));
quiver(U, V, FU, FV, 0.5)
axis equal tight, hold on
plot(UE,VE)
xlabel 'u'
ylabel 'v'
title 'Vector field and the solution for the system (d)'
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2. Classify each of the following equations as to: separable, homogeneous, exact, linear, Bernoulli, Ricatti, or Clairaut and solve.

   (a) $y' = 1/(y-x)$, (b) $y' = (x-y)/x$, (c) $(y')^2 + 2y = 2xy'$, (d) $y' = 1/x(x-y)$,

(e) $y' = (y^2 + y)/(x^2 + x)$, (f) $y' = 4 + 5y + y^2$, (g) $y\,dx = (y - xy^2)\,dy$,

(h) $xy' = y\exp(x/y) - x$, (i) $xyy' + y^2 = 2x$, (j) $2xyy' + y^2 = 2x^2$, (k) $y\,dx + x\,dy = 0$,

(l) $(x^2 + 2y/x)\,dx = (3 - \ln(x^2))\,dy$, (m) $y' = (x/y) + (y/x) + 1$,

(n) $(y/x^2)y' + \exp(2x^3 + y^2) = 0$, (o) $y = (y' - 3)^2 + xy'$, (p) $y' + 5y^2 = 3x^4 - 2xy$

3. Solve $y' = 1 + y^2$, $y(0) = 0$, by one of the usual methods. Solve the same problem by Picard's method and compare the results.
   Note : The following sequence of Matlab commands may help in Picard's method:
   ```
   syms x y %(once), y=0; %(once), y=int(1+y*y,0,x) %(repeatedly)
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4. In calculus, the curvature of a curve whose equation is $y = f(x)$ is defined to be the number $\kappa = y''/(1 + (y')^2)^{3/2}$. Determine a function for which $\kappa = 1$.
   Note : For simplicity ignore constants of integration.

5. Show that the substitutions $y = y_1 + u$ and in turn $w = u^{-1}$ reduces the Ricatti's equation $y' = P(x) + Q(x)y + R(x)y^2$, whose a particular solution is y_1, to $w' = (Q + 2y_1R)w = -R$. Classify the resulting differential equation and provide a general solution.

6. Show that $y = cx + f(c)$, where c is an arbitrary constant, is a solution of Clairaut's equation $y' = xy' + f(y')$. Show also that $F(y - cx, c) = 0$ is a family of solutions of an alternative form of Clairaut's equation $F(y - xy', y') = 0$.

7. Construct one-parameter family of solutions to the linear equation $a_1(x)y' + a_0(x)y = f(x)$ by the integrating factor method. Can you identify the particular solution in the resulting expression?

8. Suppose $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation. Show that the substitutions (i) $x = uy$ and (ii) $x = r\cos(\theta)$, $y = r\sin(\theta)$ reduce the equation to one with separable variables.

9. Show that to construct an orthogonal family ($\alpha = \pi/2$) of curves to a given family of curves $G(x, y, c) = 0$: first construct the differential equation $y' = f(x, y)$ whose solution curves are the given family $G(x, y, c) = 0$ and then solve $y' = -1/f(x, y)$ for the orthogonal family of curves. Also show that to construct an isogonal family of curves that intersect at

a specified constant angle $\alpha \neq \pi/2$, need to solve $y' = (f(x,y) \pm \tan\alpha)/(1 \mp f(x,y)\tan\alpha)$. Apply to $G(x,y,c) \equiv y - c\exp(-x) = 0$ for (i) $\alpha = \pi/2$ and (ii) $\alpha = \pi/4$.

Theorem 1: Existence of a Unique Solution
Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le x \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ and $\partial f/\partial y$ are continuous on R, then there exists an interval I centered at x_0 and a unique function $y(x)$ defined on I satisfying the initial-value problem IVP: $y' = f(x,y)$, $y(x_0) = y_0$.

10. Determine whether Theorem 1 guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point:
(a) $(1,4)$, (b) $(5,3)$, (c) $(2,-3)$, (d) $(-1,1)$

Theorem 2: Cauchy-Lipschitz
Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le x \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ is continuous and satisfy Lipschitz condition on R, then there exists an interval I centered at x_0 and a unique function $y(x)$ defined on I satisfying the initial-value problem IVP: $y' = f(x,y)$, $y(x_0) = y_0$.

Definition: Lipschitz condition
A function $f(x,y)$ is said to satisfy Lipschitz condition in R, if there exists a constant $K \ge 0$ such that $|f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1|$ for all x, y_1, y_2 in R.

11. By inspection determine a solution of the IVP: $y' = |y-1|$, $y(0) = 1$. State why the conditions of Theorem 1 does not hold for this differential equation, however, the solution to this IVP is unique.