SUPPLEMENTARY PROBLEMS

1. For each of the following systems, obtain a solution graphically, if possible, and then try to solve the system by the Gaussian elimination.

(a) $\begin{aligned} x+2y&=3, \\ x-y&=0. \end{aligned}$ **(b)** $\begin{aligned} x+2y&=3, \\ -2x-4y&=6. \end{aligned}$ **(c)** $\begin{aligned} x+2y&=3, \\ 2x+4y&=6. \end{aligned}$ **(d)** $\begin{aligned} 0\cdot x+y&=3, \\ 2x-y&=7. \end{aligned}$

Note : You may, for example in (a), use

```
ezplot('(3-x)/2',-10,10)
hold on, ezplot('x',-10,10)
```

2. We want to compare no pivoting and partial pivoting as solution strategy. Use the routines **lufact.m**, **lufactpiv.m**, the " \" operator and the "`inv()`" command to solve the systems of linear equations with the coefficient matrices A_i and the RHS vectors b_i shown below and compute the residual error $\| A_i x - b_i \|_\infty$ to compare the results in terms of how well the solutions satisfy the equation, that is, $\| A_i x - b_i \|_\infty \approx 0$.

(a) $A_1 = \begin{bmatrix} 10^{-15} & 1 \\ 1 & 10^{11} \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1+10^{-15} \\ 10^{11}+1 \end{bmatrix}$

(b) $A_2 = \begin{bmatrix} 10^{-14.6} & 1 \\ 1 & 10^{15} \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1+10^{-14.6} \\ 10^{15}+1 \end{bmatrix}$

(c) $A_3 = \begin{bmatrix} 10^{11} & 1 \\ 1 & 10^{-15} \end{bmatrix}, \quad b_3 = \begin{bmatrix} 10^{11}+1 \\ 1+10^{-15} \end{bmatrix}$

(d) $A_4 = \begin{bmatrix} 10^{14.6} & 1 \\ 1 & 10^{-15} \end{bmatrix}, \quad b_4 = \begin{bmatrix} 10^{14.6}+1 \\ 1+10^{-15} \end{bmatrix}.$

How well does the partial pivoting strategy perform in comparison to the no pivoting, the " \" operator and the "`inv()`" command?

Note : As a guide, we present the following Matlab command sequence for one case:

```
A = [1e-15 1; 1 1e11]; b = [1+1e-15; 1e10+1];
N=length(A);
% GE without pivoting
A=lufact(A); L=eye(N)+tril(A,-1); U=triu(A);
Y=forwardsolve(L,B); X=backsolve(U,Y);
res_nop = norm(A*x-b,inf);
% GE with pivoting
[A,P]=lufactpiv(A); L=eye(N)+tril(A,-1); U=triu(A);
Y=forwardsolve(L,P*B); X=backsolve(U,Y);
res_piv = norm(A*x-b,inf);
% The use of operator "\"
x = A\b; res_lsh = norm(A*x-b,inf);
% The use of inverse
x = inv(A)*b; res_inv = norm(A*x-b,inf);
```

3. Consider $AX = B$ with $A = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix}$ and $B = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}$.

(a) Compute residual vectors $\tilde{R} = B - A\tilde{X}$ and $\hat{R} = B - A\hat{X}$ for two computed solutions $\tilde{X} = [0.999 \quad -1.001]^T$ and $\hat{X} = [0.341 \quad -0.087]^T$ obtained using two different algorithms. Use relative residual based on maximum norm to decide which of $\tilde{X}$ and $\hat{X}$ is the better solution vector.

(b) Now, compute the error vectors $\tilde{E} = \tilde{X} - X$ and $\hat{E} = \hat{X} - X$, where $X = [1 \quad 1]^T$ is the exact solution. Use relative error based on maximum norm and discuss the implications of this example.

Ans.: $\text{cond}(A) = 2.7 \times 10^6$.

4. Consider the system $\begin{matrix} 10^{-4} x_1 + x_2 = b_1 \\ x_1 + x_2 = b_2 \end{matrix}$ where $b_1 \neq 0$ and $b_2 \neq 0$. Its exact solution is

$$x_1 = \frac{-b_1 + b_2}{1 - 10^{-4}} \qquad \text{and} \qquad x_2 = \frac{b_1 - 10^{-4} b_2}{1 - 10^{-4}}.$$

(a) Let $b_1 = 1$ and $b_2 = 2$. Solve this system using naive Gaussian elimination with three-digit (rounded) arithmetic and compare with the exact solution.

(b) Repeat the part (a) after interchanging the order of the two equations.

5. Consider the matrix

$$\begin{bmatrix} -0.0013 & 56.4972 & 123.4567 & 987.6543 \\ 0 & -0.0145 & 8.8990 & 833.3333 \\ 0 & 102.7513 & -7.6543 & 69.6869 \\ 0 & -1.3131 & -9876.5432 & 100.0001 \end{bmatrix}.$$

List the pivot elements and the final form of the index vector in the case when

(a) naive Gaussian elimination method,

(b) Gaussian elimination method with partial pivoting is used.

6. Let A be the $n \times n$ tridiagonal matrix

$$\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}.$$

The inverse of this matrix is known to be

$$(A^{-1})_{ij} = (A^{-1})_{ji} = \frac{-i(n+1-j)}{(n+1)} \qquad (i \le j).$$

(a) Use Doolittle method to compute the inverse of A ($n=4$) by solving $AX=B$ for suitable selections of B.

Ans.: Choose B to be the columns of I_4.

(b) Write and implement the technique in (a) as an efficient Matlab code for computing the inverse of A ($n=10$).

Note : Your code may call available Matlab routines.

7. Consider Gaussian elimination with partial pivoting applied to the coefficient matrix

$$\begin{bmatrix} \# & \# & \# & \# & 0 \\ \# & \# & \# & 0 & \# \\ 0 & \# & \# & \# & 0 \\ 0 & \# & 0 & \# & 0 \\ \# & 0 & 0 & \# & \# \end{bmatrix}$$

where each # denotes a different nonzero element. Circle the locations of elements in which multipliers will be stored and mark with an f those where fill-in (change) will occur. The final index vector is $[2 \;\; 3 \;\; 1 \;\; 5 \;\; 4]$.

8. Consider the following sequences of vectors $\{X^{(k)}\}$:

(i) $X^{(k)} = \left(1/k \quad \exp(1-k) \quad -2/k^2\right)$ (ii) $X^{(k)} = \left(e^{-k}\cos k \quad k\sin\frac{1}{k} \quad 3+k^{-2}\right)$

(a) Find $\|X^{(k)}\|_\infty$ and $\|X^{(k)}\|_1$ for a sufficiently large positive integer k.

Ans.: (i) $\|X^{(k)}\|_\infty = 1/k$, (ii) $\|X^{(k)}\|_\infty = 3+k^{-2}$.

(b) Show that these sequences are convergent, and find their limits under these norms.

Ans.: (i) $\|X^{(k)}\|_1 \to 0$, (ii) $\|X^{(k)}\|_1 \to 4$.

(c) Write and implement a Matlab code to determine their limits under the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ to 7 decimals accuracy.

Note : In a k loop, you may use `norm(X(k+1)-X(k),*)` where * stands for `inf` and `1`.

9. Find the first two iterations of (a) the Jacobi method and (b) the Gauss-Seidel method for the following linear systems

(i)
$$\begin{aligned} 2x_1 - x_2 + 10x_3 \qquad\qquad &= -11 \\ 3x_2 - x_3 + 8x_4 &= -11 \\ 10x_1 - x_2 + 2x_3 \qquad\qquad &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \end{aligned}$$

(ii)
$$\begin{aligned}
10x_1 - x_2 + 2x_3 \qquad &= 6 \\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\
2x_1 - x_2 + 10x_3 \qquad &= -11 \\
3x_2 - x_3 + 8x_4 &= -11
\end{aligned}$$

(c) Write and implement these iterative techniques in a Matlab code to solve the linear systems to three decimals accuracy. Set the maximum number of iterations to $N\max = 25$.

(d) Noting that the systems in (i) and (ii) are the same but arranged differently, how can you explain the resulting behavior.

Ans.: (ii) is strictly diagonally dominant so the convergence is guaranteed.

10. Find the first two iterations of the Newton's method applied to the following nonlinear system $\begin{matrix} 3x^2 - y^2 = 0 \\ 3xy^2 - x^3 - 1 = 0 \end{matrix}$ starting from an initial approximation $X^{(0)}$ obtained graphically.

Note : Here is a Matlab code segment to obtain graphical solution as the intersection of contours

```
[X,Y] = meshgrid(-2:.2:2,-2:.2:2);
Z1=3*X.^2-Y.^2; Z2=3*X.*Y.^2-X.^3-1;
contour(X,Y,Z1,[0 0],'r-');
hold on, contour(X,Y,Z2,[0 0],'b-');
```