NUMERICAL SIMULATION OF LONG WAVE RUNUP ON A SLOPING BEACH*

by

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ABSTRACT

The one-dimensional nonlinear shallow-water wave equations over a physical domain of linearly sloping beach are numerically integrated using a Chebyshev collocation spectral method. A domain decomposition technique is employed in order to facilitate the numerical implementation of the method. Interface conditions and open-water transparent boundary conditions are imposed. The moving boundary due to the shoreline motion is treated exactly by a front-fixing technique. The performance of the numerical method in prediction of the long wave propagation and runup on a sloping beach is successfully tested against analytic/semi-analytic results available in literature.

Keywords : Runup, Nonlinear shallow-water wave equations, Spectral method.

1. INTRODUCTION

The runup of tidal waves or tsunamis generated by impulsive geophysical events are known to cause extensive inundation and loss of life. Faced with such catastropic events, the prediction of runup heights and understanding the mechanisms of inundation have been an important research area. The nonlinear inviscid shallow-water wave equations have been widely used in modeling of the propagation of long waves such as an incoming tsunami. Long waves of various profiles are used both experimentally and numerically as models for the far-field tsunamis in literature. Their evolution and runup on a sloping beach under the shallow-water equations are studied numerically and analytically to provide insight into the mechanisms of tsunami inundation [1, 2, 3, 4, 5, 6].

The main difficulty of the numerical simulation of this problem is due to the moving boundary associated with the shoreline motion. This issue of the moving boundary is addressed in literature [1, 3, 6] by employing various techniques, such as stepwise approximation of sloping beach in which the shoreline is located between wet and dry finite difference cells. We treated the moving boundary by a front-fixing technique [7] in which the moving boundary of the computational domain is replaced by a stationary boundary using a transformation.

In this work, we use a Chebyshev collocation spectral method as the numerical solver. Unlike finite difference methods, the spectral methods are global methods which uses

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expansion of the unknown function in terms of basis functions globally defined in the computational domain and under suitable conditions achieve very high (spectral) accuracy [8, 9]. The absence of smoothness in the data and/or the solution adversely effects the accuracy of the method. A domain decomposition technique is implemented in order to achive piecewise smoothness in each subdomains as well as to keep the size of the individual subdomains small. Interface conditions are imposed at the interfaces to allow information transfer between the subdomains. The open-water boundary is made transparent to outgoing disturbances.

The performance of the numerical method in the prediction of the long wave propagation and runup on a linearly sloping beach is tested against two analytic/semi-analytic studies [4, 5]. Synolakis [5] analytically solved the linear shallow-water wave equation for solitary wave propagation over the canonical bathymetry, i.e. a linearly sloping beach connected with a constant-depth segment and provided analytical expression for the maximum runup. Carrier *et al.* [5] performed a semi-analytic study of long wave propagation and runup under the nonlinear shallow-water wave equations on a uniformly sloping beach and provided the four cases of initial waveforms having Gaussian and leading-depression N-wave shapes as examples.

2. FORMULATION

2.1 THE GOVERNING EQUATIONS

We consider one-dimensional nonlinear shallow-water equations in differential conservation form [10]

$$\begin{aligned} &H_t + (Hu)_x = 0, \\ &(Hu)_t + \left(Hu^2 + \tfrac{1}{2} gH^2\right)_x = -gHb_x, \end{aligned} \tag{1}$$

where $u(x,t)$, $H(x,t)$, $b(x)$, and g are the horizontal depth-averaged velocity, the water depth, the bed profile, and the gravitational constant, respectively. In this work, we study the special case of linearly sloping bed so that $b(x) = \alpha x$ and $H(x,t) = h_0 - \alpha x + \eta(x,t)$ where h_0 is the still water depth at the seaward boundary and $\eta(x,t)$ is the water surface heigth above the still level.

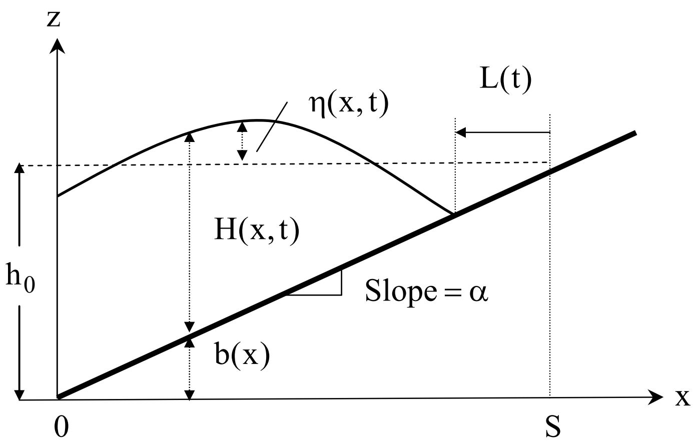

Figure 1. The geometry of the physical domain.

2.2 FRONT-FIXING TRANSFORMATION [7]

The governing system of equations (1) is considered inside the domain $0 \le x \le S - L(t)$. Here, $L(t) < S$ is the time-dependent distance from the shoreline to the point where the total depth of the water vanishes and it is defined by

$$\alpha L(t) = -\eta(S - L(t), t).$$

Note that for the runup, $L(t)$ is negative, while for the draw-down, it is positive in this formulation.

Front-fixing transformation amounts to transforming the moving boundary of the domain $x \in [0, S - L(t)]$ into the stationary domain $\zeta \in [0,1]$. This is accomplished by defining a new set of independent variables,

$$\zeta = \frac{x}{S - L(t)} \qquad \text{and} \qquad t = t.$$

Further, the dependent variables, H and u, following Gelb *et al.* [11], are transformed using

$$\omega_1(\zeta, t) = H(x,t)(S - L(t)),$$
$$\omega_2(\zeta, t) = H(x,t)(u(x,t) + g\alpha t)(S - L(t)).$$

Under these transformations of the dependent and independent variables, the governing equations take the following form after some algebra;

$$\begin{aligned} &\frac{\partial \omega_1}{\partial t} + \frac{1}{(S - L(t))}\frac{\partial}{\partial \zeta}\left[\left(\frac{dL}{dt}\zeta - g\alpha t\right)\omega_1 + \omega_2\right] = 0, \\ &\frac{\partial \omega_2}{\partial t} + \frac{1}{(S - L(t))}\frac{\partial}{\partial \zeta}\left[\left(\frac{dL}{dt}\zeta - g\alpha t\right)\omega_2 + \frac{\omega_2^2}{\omega_1} + \frac{1}{2}g\frac{\omega_1^2}{(S - L(t))}\right] = 0. \end{aligned} \tag{2}$$

In particular, the boundary condition at $\zeta = 1$ in this formulation becomes

$$\omega_1(1,t) = \omega_2(1,t) = 0, \tag{3}$$

that, in turn, produces the equation

$$\frac{dL}{dt} = g\alpha t - \left.\frac{\partial \omega_2 / \partial \zeta}{\partial \omega_1 / \partial \zeta}\right|_{\zeta = 1}. \tag{4}$$

The Riemann invariants for the transformed hyperbolic system (2) are

$$r^{\pm} = \frac{\omega_2}{\omega_1} \pm 2\sqrt{\frac{g\omega_1}{(S - L(t))}},$$

whereas, those of the original governing equations (1) are $s^{\pm} = u \pm 2\sqrt{gH}$.

3. NUMERICAL IMPLEMENTATION

3.1 NUMERICAL METHOD

In this work, we use a Chebyshev collocation spectral method as the numerical solver. Unlike finite difference methods, the spectral methods are global methods which use expansion of the unknown function in terms of basis functions globally defined in the computational domain and, under suitable conditions of smoothness in the data and/or the solution, achieve very high accuracy [8]. For the Chebyshev collocation method, a function $f(x,t)$ is approximated by

$$f_N(x,t) = \sum_{k=0}^{N} a_k(t) T_k(x),$$

in the interval $[-1,1]$ where the basis functions are defined by

$$T_k(x) = \cos\big((k-1)\arccos x\big),$$

and the coefficients $a_k(t)$ are evaluated based on the interpolation condition [9] at the collocation points

$$x_j = \cos\tfrac{\pi(j-1)}{(N-1)}, \quad j = 1, 2, \ldots, N.$$

The governing equations are then forced to accept $f_N(x,t)$ as the solution at the collocation points resulting in N semi-discrete equations to be integrated in time for $a_k(t)$.

Figure 2. The placement of the Chebyshev grid (collocation) points.

3.2 BOUNDARY AND INTERFACE CONDITIONS

The Chebyshev collocation points are placed more densely at the boundaries, $O(N^{-2})$, than in the interior, $O(N^{-1})$, of its domain. While this facilitates the accurate imposition of the boundary conditions, it also severely restricts the size of the time step used in the numerical integration to $O(N^{-2})$ for stability reasons. The sizes and thus the resolution requirements, N, of the individual subdomains are, therefore, kept small by implementing a domain decomposition technique. The interfaces of the subdomains are placed so that smoothness of the data of the problem, such as the bed profile, is achieved within the subdomains.

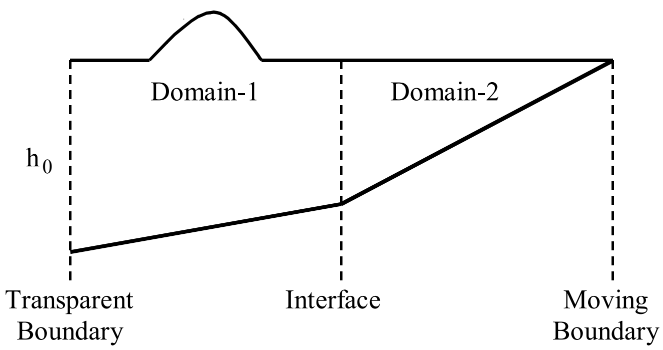

Figure 3. Boundary and interface configurations.

In a hyperbolic problem, the information across the domain is carried by the Riemann invariants along the characteristics. For this purpose, the computed left-going invariant r_2^- in the domain-2 and the right-going invariant r_1^+ in the domain-1 are matched at the interface to determine the interface conditions for the dependent variables. This procedure is also repeated at the open-water boundary for transparency where the right-going invariant is taken as $s^+ = 2c_0 = 2\sqrt{gh_0}$ which signifies no incoming information.

4. TEST CASES

We conducted two different sets of numerical experiments to test the implementation of the numerical method and the performance of the open-water transparent boundary and interface conditions. The first case will be a solitary wave propagation over the canonical bathymetry, i.e. a linearly sloping beach connected with a constant-depth segment and the second will be long wave propagation and runup on a uniformly sloping beach.

4.1 CASE I

Synolakis [5] analytically solved the linearized form of the equations (1) for solitary wave propagation over the canonical bathymetry and provided an analytical expression for the maximum runup, referred to as runup law. In order to prepare grounds for comparison, we select a solitary wave profile

$$\eta(x,0) = A \operatorname{sech}^2(\gamma(x - x_c)),$$

where A is the wave amplitude and $\gamma = \sqrt{3A/4}$. The wave is centered at x_c where is half-wavelength, $W/2 = \operatorname{arccosh}\sqrt{1/0.05}/\gamma$, away from the toe of the sloping beach at the constant depth segment of the bathymetry. The initial velocity is taken as

$$u(x,0) = 2\sqrt{gH(x,0)} - 2\sqrt{g(H(x,0) - \eta(x,0))},$$

in order to prevent the initial disturbance to produce an outgoing wave. The computational domain is decomposed into the constant-depth (domain-1) and the linearly sloping beach (domain-2) segments. In domain-1, the open-water transparent boundary and the interface conditions, whereas, in domain-2, the interface and the moving boundary conditions are imposed. While the equations (1) are numerically integrated in time for domain-1, the equations (2-4) are used for domain-2. The resulting propagation, the shoreline motion, and the maximum runup heights are shown in figures 4 and 5.

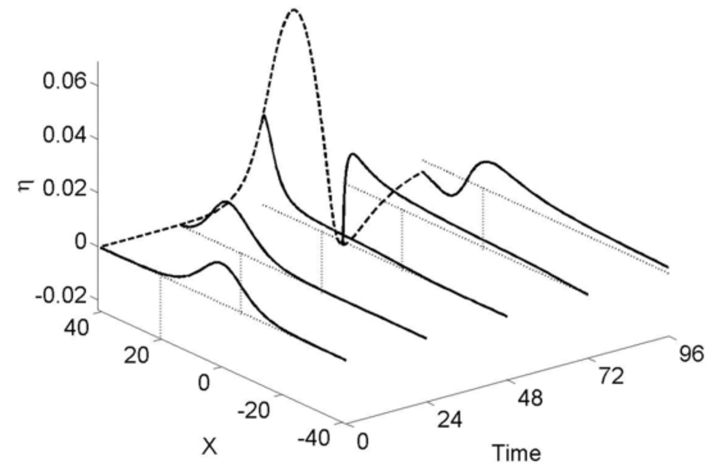

Figure 4. Solitary wave propagation and runup over the canonical bathymetry with $\alpha = 1:20$ and $A = 0.015$. Here, the dash-line shows the runup, the solid-lines are the spatial variations of the water-surface elevation at different nondimensional time ($= t/\sqrt{h_0/g}$) values. The vertical dotted-line illustrates the interface, while the horizontal dotted-line is the still water surface.

The propagation picture in figure 4 shows that the unsplitting initial waveform propagates towards the beach, reaches its maximum runup and then reflects back towards the open water. The conditions imposed at the interface and the open-water boundary are performing successfully. The comparison between the runup law and our numerical runup values satisfactorily agree, as shown in figure 5, for the few slope and initial wave height values that we run.

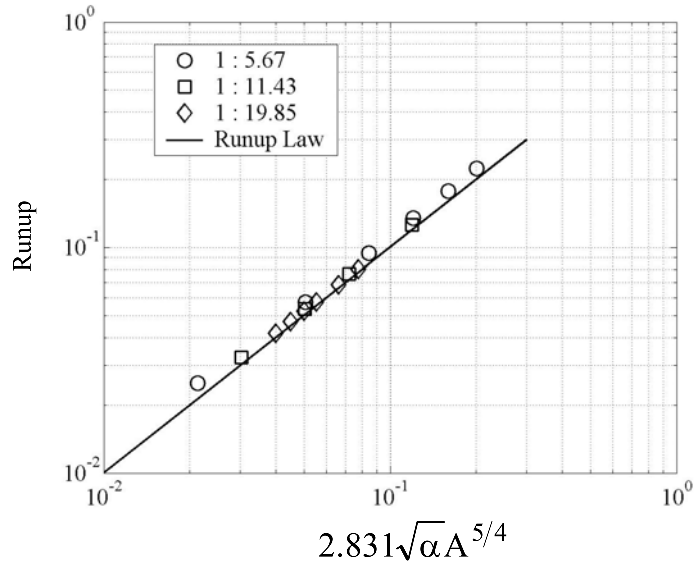

Figure 5. Runup versus the runup law [4].

4.2 CASE II

Carrier *et al.* [5] performed a semi-analytic study of long wave propagation and runup/draw-down under the nonlinear equations (1) on a uniformly sloping beach for the four cases of initial waveforms having the Gaussian and leading-depression N-wave shapes. These are:

(a) $\eta(x,0) = 0.017\exp\{-4(x-1.69)^2\}$,

(b) $\eta(x,0) = -0.017\exp\{-4(x-1.69)^2\}$,

(c) $\eta(x,0) = 0.02\exp\{-3.5(x-1.5625)^2\} - 0.01\exp\{-3.5(x-1.0)^2\}$, and

(d) $\eta(x,0) = 0.006\exp\{-0.4444(x-4.1209)^2\} - 0.018\exp\{-4(x-1.6384)^2\}$.

The initial velocity is taken as nil. We have numerically integrated the equations (2-4) in time for the given cases over the linearly sloping beach having the moving boundary to the right and the open-water boundary to the left. The resulting propagation and the shoreline motion is shown in figure 6 for each case.

Figure 6 shows the successful performance of the open-water transparent boundary condition as the outgoing piece of the splitting initial disturbance disappears into the open-water. Table 1 satisfactorily compares the extreme values of the runup and the draw-down.

Table 1. Comparison of the extreme values of the runup and the draw-down.

	Cases	(a)	(b)	(c)	(d)
Maximum runup	Numerical	- 0.0467	- 0.0267	- 0.0575	- 0.0327
	Carrier *et al.* [5]	- 0.0470	- 0.0268	- 0.0583	- 0.0328
Maximum draw-down	Numerical	0.0277	0.0471	0.0247	0.0480
	Carrier *et al.* [5]	0.0268	0.0470	0.0235	0.0484

5. CONCLUSIONS

The results presented indicate that the formulation and the numerical implementation of the problem perform satisfactorily. The open-water boundary condition provides the intended transparency to the outgoing waves. The interface conditions are successful in transferring information across. The numerically challenging problem of moving boundary is handled satisfactorily by the front-fixing transformation implemented.

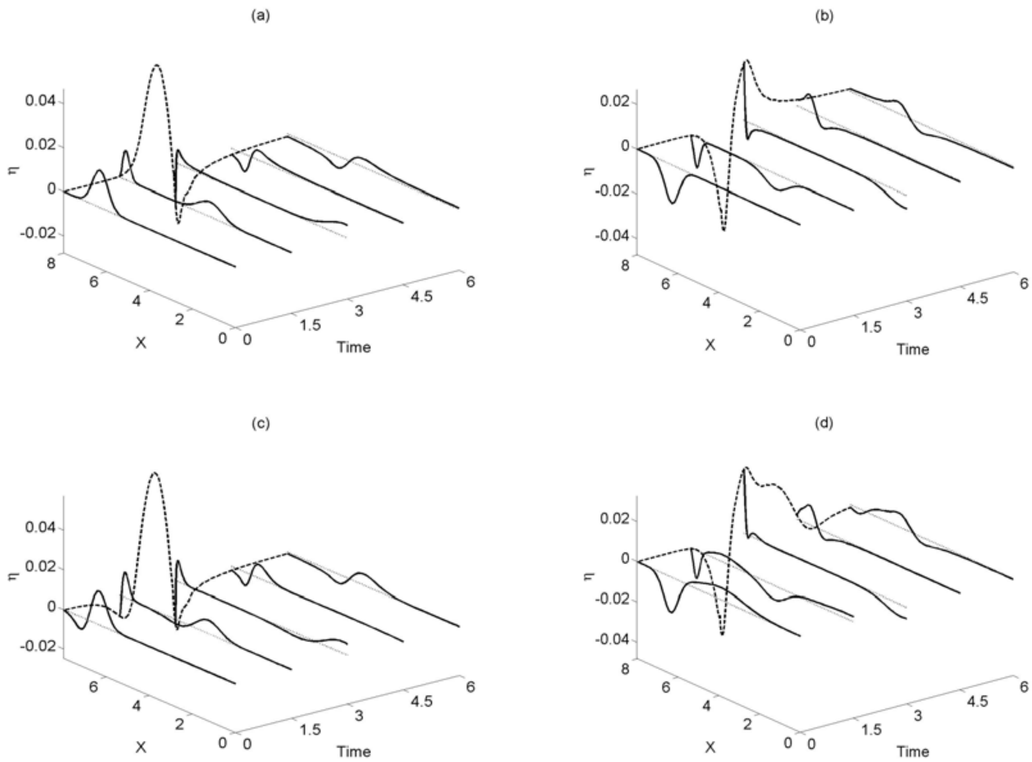

Figure 6. Wave propagation and shoreline motion in four cases of Carrier *et al.* [5].

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