Midterm 1, Part 1

April 8, 50 points in 4 questions

Name and the student number:

Q1. (16 pts) Explain briefly the following notions, and state the corresponding claims.

(1) What is a critical point of a map $f : M \rightarrow N$ ? What is a critical value of $f$ ?

(2) What is an atlas of class $C^r$ on a manifold ?

(3) What is a differential structure of class $C^r$ on a manifold ? How it can be defined?

(4) State Sard’s theorem.

(5) What is a smooth isotopy of diffeomorphisms ?

(6) What is a Lie group ?

(7) What does mean “homogeneity of manifolds” (state the property) ?

(8) What is the induced orientation on the boundary of an orientable manifold ?
Q2. (2 pts) Under which conditions on manifolds $X, Y$ and $Z$ the degrees $\deg f, \deg g, \deg g \circ f$ for smooth maps $f: X \to Y$ and $g: Y \to Z$ are well-defined as integers?

Q3. (8 pts) For the map $h: \mathbb{R}^2 \to \mathbb{RP}^2$, $h(x, y) = [x : y^2 : x^2 y]$, find the critical points and the critical values.
Q4. (24 pts)

(1) Show that chart \( f : \mathbb{R} \to \mathbb{R}, f(x) = 2x, \) defines the standard differential structure, \( C_{st}, \) in \( \mathbb{R}. \)

(2) Show that chart \( g : \mathbb{R} \to \mathbb{R}, g(x) = \begin{cases} x, & \text{if } x \leq 0 \\ 2x, & \text{if } x \geq 0 \end{cases} \) defines a non-standard differential structure, \( C_g, \) in \( \mathbb{R}. \)

(3) Check if \( h : \mathbb{R} \to \mathbb{R}, h(x) = x^3, \) is a differentiable function with respect to the differential structure \( C_g. \) Find its class \( C^r. \)

(4) Check if \( f \) and \( g \) are differentiable with respect to the smooth structure \( C_h \) defined by chart \( h. \)