Q1. (10 pts)

(1) What does mean “map \( f: M \to N \) is transversal to a submanifold \( L \subset N \)” ?

(2) State the theorem about the preimage of a submanifold with respect to a map transversal to that submanifold.

(3) What is the exponential map ?

(4) What is a tubular neighborhood of a submanifold ?

(5) What is a smooth homotopy ?

(6) What is a Riemannian metric in a vector bundle \( E \) ? How it can be defined as a tensor field ?

(7) State the Whitney embedding theorem.

(8) What is a cobordism from manifold \( M_0 \) to manifold \( M_1 \) ? In which case it is called “oriented” ?
Q2. (8 pts) Give a more simple description of manifolds obtained by the following constructions.
(a) The double of $S^2 \times S^3$.
(b) The connected sum $\mathbb{C}P^2 \# S^4$.
(c) The connected sum of two tori $T^2$.
(d) The double of the Möbius band.
(e) A tubular neighborhood of $S^1$ in $\mathbb{R}^2$.
(f) A tubular neighborhood of a knot ($S^1$ embedded in $\mathbb{R}^3$).

Q3. (8 pts) Explain why any closed surface embedded into $\mathbb{R}^3$ is orientable.

Q2. (8 pts) (a) Map $f$ from 15-dimensional manifold $M$ to $\mathbb{R}^{20}$ is transverse to a 13-dimensional submanifold $L \subset \mathbb{R}^{20}$. What is the dimension of $f^{-1}(L)$?
(b) For a given homotopy $X \times [0,1] \to \mathbb{C}P^2 \times S^5$, the preimage $f^{-1}(y)$ of a regular value $y$ is $\mathbb{R}P^2$. Find the dimension of $X$.
(c) In the previous question, show that $X$ cannot be orientable.

Q3. (8 pts) (a) Show that a sphere $S^{n-1}$ in $\mathbb{R}^n$ has a trivial normal bundle.
(b) Knowing that $S^3$ is parallelizable (which means that its tangent bundle $TS^3$ is trivial) show that its self-intersection in $TS^3$ is zero.
(c) Show that for any parallelizable manifold $X$ its Euler characteristic $\chi(X)$ vanishes.
(d) Knowing that $\chi(X) = 0$ deduce that the self-intersection index of the diagonal $\Delta_X \subset X \times X$ is zero.
(e) Find the self-intersection index of the diagonal $\Delta_F \subset F \times F$ where $F$ is a closed oriented surface of genus 2.

Q5. (10 pts) Knowing that a vector field $V$ in $\mathbb{C}P^2$ has only non-degenerate zeros, estimate from below the number of zeros.

Q5. (10 pts) (a) Map $f: X \to Y$ is null-homotopic and $g: X \to Z$ is any map, $\dim X = \dim Y + \dim Z = n$. Prove that the map $X \to Y \times Z$, $x \mapsto (f(x), g(x))$ has degree 0.
(b) Show that any map $S^n \to S^1 \times X$, $\dim X = n - 1$, has degree 0.