$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

Lecture 7. Late 17th and 18th century

Bernoulli brothers

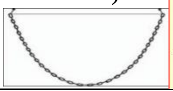
On the Jacob Bernoulli tomb: "*Eadem Mutata resurgo*" I shall arise the same, though changed

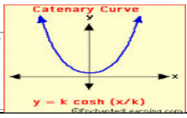
Jacob Bernoulli (1654–1705) *The Art of Conjecture*

1713: probability theory, expected values, permutations and combinations, *Bernoulli trials* and *distribution*, *Bernoulli Numbers* and

$$\frac{t}{e^t-1}=\sum_{m=0}^{\infty}\mathrm{B}_m\frac{t^m}{m!}$$

polynomials, proved the Law of Large Numbers. He invented polar coordinates, first used word "integral" for the area under a curve and convinced Leibniz to change the name of the new math from calculus sunmatorius to calculus integralis. He found the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$ (notation "e" was proposed later by Euler). He studied differential equations, like "Bernoulli equations". $y'=p(x)y+q(x)y^n.$ Transcendental curves: the catenary (hanging chain), the brachistocrone (shortest time) leaded to Calculus of variations.

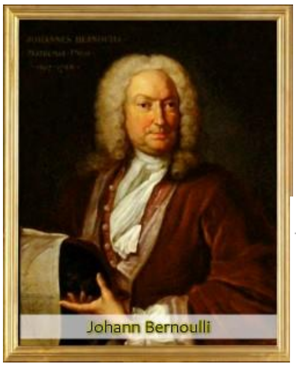




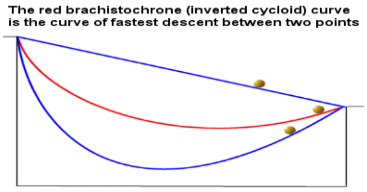
Nicolaus Bernoulli , brother of Jacob-Johann, 1662–1716 painter and alderman of Basel

Nicolaus I Bernoulli (1687–1759) son of Nicolaus, thesis on probability theory in law. 1716 Galileo chair in Padua, dif.equations and geometry, 1722 Chair in Logic at Basel University, stated in his letters St.Petersburg Paradox, in 1712-14 discussed divergence of $(1+x)^{-n}$ 1742-43 in a letter to Euler criticized his solution of Basel Problem, found error in Newton's treating of higher derivatives.

Johann Bernoulli (1667–1748) student - Leonhard Euler



Paris Academy's biennial prize winner in 1727, 1730, and 1734, gave private calculus lessons to Marquis de L'Hôpital in 1691, who published a book in 1696



based on Johann's lectures containing l'Hôpital's Rule (that was found by Johann).

1696: Brachisrtocrone problem stated in Acta Eruditorum by Johann and solved independently by Newton, Leibniz, Jakob Bernoulli, Tschirnhaus and l'Hôpital.

Invented analytical trigonometry.

In 1738 plagiarized from his son Daniel by writing a book *Hidraulica* in which the date was shown 2 years earlier than it was written

$$1^k+2^k+3^k+\cdots+(n-1)^k=\frac{1}{k+1}\Big\{\binom{k+1}{0}B_0\,n^{k+1}+\binom{k+1}{1}B_1\,n^k+\binom{k+1}{2}B_2\,n^{k-1}+\cdots+\binom{k+1}{i}B_i\,n^{k-i+1}+\cdots+\binom{k+1}{k}B_k\,n\Big\}$$

$$B_0=1\quad B_1=-\frac{1}{2}\quad B_2=\frac{1}{6}\quad B_4=-\frac{1}{30}\quad B_6=\frac{1}{42}$$

$$B_8=-\frac{1}{30}\quad B_{10}=\frac{5}{66}\quad B_{12}=-\frac{691}{2730}\quad B_{14}=\frac{7}{6}\quad \text{etc.}$$

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\frac{B_1x^2}{2!}-\frac{B_2x^4}{4!}+\frac{B_3x^6}{6!}-\cdots\quad |x|<2\pi$$

$$1-\frac{x}{2}\cot\left(\frac{x}{2}\right)=\frac{B_1x^2}{2!}+\frac{B_2x^4}{4!}+\frac{B_3x^6}{6!}+\cdots\quad |x|<\pi$$

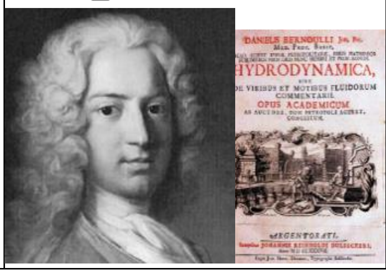
Children of Johann

Nicolaus II Bernoulli (1695–1726) curves, dif. equations, probability, fluid dynamics, 1719: Chair in Math in Padua, posed the problem of reciprocal orthogonal trajectories, 1725: St.Petersburg, died after 8 months from appendicitis.

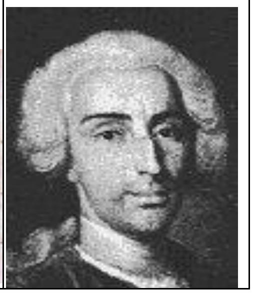
Daniel Bernoulli (1700–1782) friend of Euler, probability (mortality statistics: efficiency of vaccination), economics: risk aversion and risk premium, 1738 fluid mechanics and kinetic theory of gases in *Hydrodynamica*, vibrating string

$$P+\frac{1}{2}\rho v^2=\text{constant}$$

(Pressure, Fluid density, Particle velocity)



Johann II Bernoulli (1710–1790) propagation of heat and light, magnets, won Paris Academy Prize 4 times, appointed to his father's chair at Basel



The Doctrine of Chances was the second book on probability theory (the first was Cardano's): discussed the normal distribution; de Moivre was the first to postulate the Central Limit Theorem, and to prove results on the Poisson distribution; de Moivre's formula $(\cos x+i\sin x)^n=\cos(nx)+i\sin(nx).$ relates trigonometry and calculus. De Moivre gave Binet's formula for Fibonacci numbers F_n

$$F_n=\frac{\phi^n-(-\phi)^{-n}}{\sqrt{5}}=\frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}$$

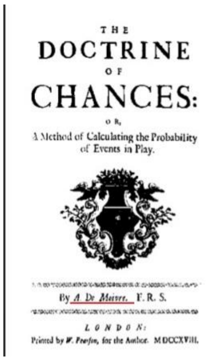
1733 de Moivre gave "Stirling formula" ***n! = cn***$^{n+1/2}e^{-n}$ and approximated constant ***c***, while James Stirling found that ***c*** was $\sqrt{2\pi}$

Moivre.

Abraham de Moivre, known for his textbook on probability theory, predicted his own death. He became more lethargic as he aged, and noticed he was sleeping an extra 15 minutes each night. He calculated that he'd die on the day his additional sleep time added up to 24 hours, which was November 27th, 1754. He was right.

Born 1667 in Champagne, France
Died 1754 in London, England

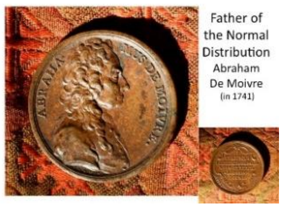
De Moivre had hoped for a chair of mathematics, but foreigners were at a disadvantage, so although he was free from religious discrimination, he still suffered discrimination as a Frenchman in England.



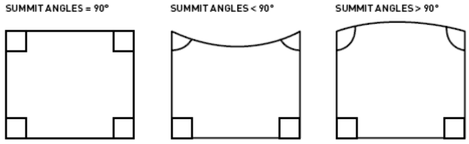
Stirling's Inequality for $n!$

$$\sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n+1}}<n!<\sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n}}$$

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$



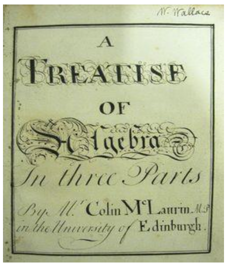
Father of the Normal Distribution Abraham De Moivre (in 1741)



Giovanni Girolamo Saccheri (1667-1733) Italian Jesuit priest, scholastic philosopher, and mathematician primarily known for a publication in 1733 related to non-Euclidean geometry (similar to the 11th Century work of Omar Khayyám which was ignored until recently). The Saccheri quadrilateral is now sometimes referred to as the Khayyam-Saccheri quadrilateral.

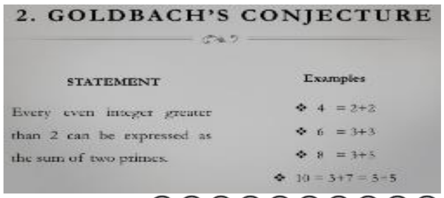
George Berkeley (1685-1753), Bishop**,** philosophic theory of "immaterialism" or "subjective idealism", 1734 *The Analyst*, with critics of the foundations of calculus.

Brook Taylor (1685-1731) best known for Taylor's theorem and the Taylor series; in book *Methodus Incrementorum Directa et Inversa* (1715) added to math a new branch "calculus of finite differences", invented integration by parts, and stated Taylor's expansion (which was actually used by James Gregory yet, but remained unrecognized till 1772, when Lagrange proclaimed it the basic principle of dif. Calculus). In the same year of 1715 an essay *Linear Perspective.*


A TREATISE OF Algebra In three Parts By M^r Colin M^cLaurin in the University of Edinburgh

Colin Maclaurin (1698-1746) a special case of the Tailor series was called Maclaurin series because it was used extensively by Maclaurin (but it was not his discovery); Euler–Maclaurin formula, derived Stirling's formula, integration formulas including Simpson's rule. 1748 *Treatise of Algebra*: linear systems with 2 and 3 unknowns preceded by two years Cramer's publication.

Christian Goldbach (1690-1764) Prussian mathematician who also studied law and worked in Russia since 1725, friend of Euler, in a letter to him in 1742 he stated Goldbach's conjecture. In 1737 one of two heads of Russian Acad. Sci., in 1740 moved from Academy to Ministry of Foreign Affairs, got lands and high noble status.

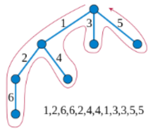

2. GOLDBACH'S CONJECTURE

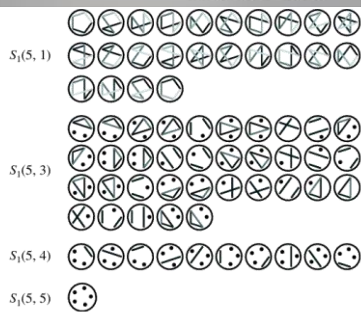
STATEMENT: Every even integer greater than 2 can be expressed as the sum of two primes.

Examples:
- 4 = 2+2
- 6 = 3+3
- 8 = 3+5
- 10 = 3+7 = 5+5

James Stirling (1692-1770 Edinburgh) Stirling numbers, Stirling permutations, and Stirling's approximation; proved the correctness of Newton's classification of cubics; studied convergence of infinite products, Γ-function

$$
\begin{array}{ccccccccccc}
 & & & & & 1_{[1]} & & & & & \\
 & & & & 1_{[1]} & & 1_{[2]} & & & & \\
 & & & 1_{[1]} & & 3_{[2]} & & 2_{[3]} & & & \\
 & & 1_{[1]} & & 7_{[2]} & & 12_{[3]} & & 6_{[4]} & & \\
 & 1_{[1]} & & 15_{[2]} & & 50_{[3]} & & 60_{[4]} & & 24_{[5]} &
\end{array}
$$





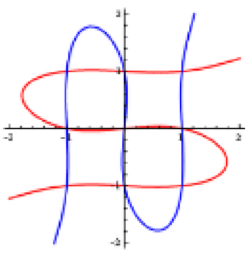
Thomas Bayes (1701-1761) English statistician, philosopher; Bayes' theorem (edited and published after his death by Richard Price.

$$p(B|A) = \frac{p(A|B)p(B)}{p(A)}$$

Gabriel Cramer (1704-1752) from Geneva. 1750 treatise on algebraic curves: proved that a curve of the *n*-th degree is determined by *n*(*n* + 3)/2 generic points on it; Cramer's rule for

Cramer's paradox: curves of degree *d* are determined by d(d+3)/2 points, but two such curves intersect at d^2 points; was resolved by Euler, who noticed that the corresponding linear systems is overdetermined.

$$\begin{cases} ax + by + cz &= j \\ dx + ey + fz &= k \\ gx + hy + iz &= l \end{cases}$$



$$d^2 - \left(\frac{d(d+3)}{2} - 1\right) = \frac{(d-1)(d-2)}{2}$$

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}.$$

Leonhard Euler's life

April 15, 1707 – September 18, 1783

Switzerland

Basel

Located in north-west Switzerland, at the border with Germany and France, on the river Rhine. Basel functions as a major industrial centre for the chemical and pharmaceutical industry.

It is an international city, with the headquarters of worldwide Swiss producers e.g. Roche, Novartis, Ciba, etc, Swiss transport e.g. Panalpina, Danzas, etc. It is where the headquarters of the international Bank for International Settlements are.

It has the oldest university of the Swiss Confederation (1460), where the famous mathematicians Jacob I Bernoulli, Jacob II Bernoulli, Johann I Bernoulli, Johann II Bernoulli, Johann III Bernoulli, Daniel Bernoulli, Nicolaus I Bernoulli, Nicolaus II Bernoulli, Euler, Hermann, studied or taught.

In the 18th century, Basel was already one of the largest cities in Europe.

The Bernoulli's

Jacop I Bernoulli (above) was a Swiss mathematician and Professor at the University of Basel. He is remembered, amongst others, for his work in probability and his introduction of the law of large numbers. His brother, Johann Bernoulli, was also a famous mathematician; he identified Euler and gave him private lessons. His son Daniel Bernoulli and Euler were the first to put together a useful theory of elasticity.

Leonhard Euler (pronounced "Oiler") (Basel, Switzerland, April 15, 1707 - St. Petersburg, Russia, September 18, 1783) was a Swiss mathematician and physicist, considered to be the preeminent mathematician in history; he is also listed on the Guinness Book of Records as the most prolific, with collected works filling about 80 quarto volumes.

Euler developed important concepts and established mathematical theorems in fields as diverse as calculus, number theory, topology, etc. He introduced the fundamental notion of a mathematical function, and set much of the modern mathematical terminology and notation.

Childhood

Euler spent most of his childhood around Basel. His father was a pastor and a family friend of the Bernoullis. Johann Bernoulli, who was then regarded as Europe's foremost mathematician, would eventually be an important influence on Euler. At the age of 13, Euler matriculated at the University of Basel, and graduated two years later. At this time, he was receiving Saturday afternoon lessons from Johann Bernoulli who quickly discovered his new pupil's incredible talent for mathematics.

His father wanted him to become a pastor. Johann Bernoulli intervened, and convinced that Euler was destined to become a great mathematician. Euler, at the age of 20, entered the Paris Academy competition, where the problem that year was to find the best way to place the masts on a ship. He won second place but eventually won the coveted annual prize 12 times in his career.

St. Petersburg

Around this time Johann Bernoulli's two sons, Daniel and Nicolas were working at the Imperial Russian Academy of Sciences in St Petersburg. In 1726, Daniel recommended that the post in physiology be filled by Euler and he accepted the offer.

He was then immediately promoted to a position in the mathematics department. He lodged with Daniel Bernoulli with whom he often collaborated. Euler mastered Russian and settled into life in St Petersburg.

The Academy at St. Petersburg was established by Peter the Great and was intended to improve education and science in Russia. As a result, it was very attractive for foreign scholars like Euler, thanks to financial resources and a comprehensive library. Euler swiftly rose through the ranks and was made professor of physics in 1731. In 1733, Euler succeeded Daniel Bernoulli as the head of the mathematics department.

In 1734, Euler got married. The young couple bought a house by the River Neva, and had thirteen children, of whom only five survived childhood.

EPFL (Swiss Federal Institute of Technology, Lausanne)'s mathematics library

Berlin

Frederick the Great of Prussia offered him a post at the Berlin Academy, which he accepted. He left St. Petersburg in 1741 and lived 25 years in Berlin, where he wrote over 380 articles. In Berlin, he published the two works which he would be most renowned for: the Introductio in analysin infinitorum and the Institutiones calculi differentialis.

In addition, Euler was asked to tutor Frederick's niece. He wrote over 200 letters to her, which were later compiled into a best-selling volume, all across Europe and in America. This work contained Euler's research physics and mathematics, as well as a valuable insight of his personality and religious beliefs. The popularity of the Letters testifies to Euler's ability to communicate scientific matters effectively to a lay audience, a rare ability for a dedicated research scientist.

Return to Russia

At the age of 59, Euler accepted an invitation to return to the St. Petersburg Academy.

Euler had to overcome several tragedies in his second stay. A fire in St. Petersburg cost him his home and almost his life. In 1773, he lost his wife of 40 years, and remarried three years later.

In 1783, he suffered a brain hemorrhage and died. His eulogy was written for the French Academy by the Marquis de Condorcet, and an account of his life, with a list of his works, by Nikolaus Fuss, Euler's grandson-in-law and the secretary of the Imperial Academy of St. Petersburg. The mathematician and philosopher Marquis de Condorcet commented,

"...il cessa de calculer et de vivre," (he ceased to calculate and to live).

A mezzotint based on the Johann George Brucker's portrait from 1735.

A 1753 portrait by Emanuel Handmann. This portrait suggests problems of the right eyelid and that Euler is perhaps suffering from strabismus. The left eye appears healthy, as it was a later cataract that destroyed it. He compensated for it with his mental calculation skills and photographic memory. Euler could repeat the Aeneid of Virgil from beginning to end without hesitation, and indicate the first and last line of every page in the book he used.

A 19th century engraving by Benjamin Holl after a portrait by Antonio Maria Lorgna.

$$e^{i\pi} + 1 = 0$$

$e^{ix} = \cos x + i \sin x$
Euler's discoveries

Euler worked in almost all areas of mathematics: geometry, calculus, trigonometry, algebra, and number theory, not to mention continuum physics, lunar theory, etc. His importance in the history of mathematics cannot be overstated: his works correspond to about 80 quarto volumes.

Mathematical notation

Euler introduced and popularized several notational conventions, including the concept of a functions: f(x), cos(x), sin(x), etc. He introduced the letters e for the base of the natural logarithm (Euler's number), Σ for sums and i ($\sqrt{-1}$). He popularized π (3.1415...).

Analysis

Thanks to his friends Bernoullis, Euler focused on calculus, at the forefront of 18th century, and made key contributions. He frequently used the logarithm function as a tool and expressed logarithmic functions in terms of power series.

He is also well known for his frequent use and development of power series i.e. functions as infinite sums of powers of the variable, such as $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
As an other example, he used $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{5!} + \cdots$ to solve for the first time the famous Basel Problem: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) = \ldots \pi^2/6$.

Euler introduced the use of the exponential function and logarithms in analytic proofs. He defined the exponential function for complex numbers and discovered its relation to the trigonometric functions. For any real number x, Euler's formula is $e^{ix} = \cos x + i \sin x$ which implies "the most remarkable formula in mathematics" (R. Feynman), Euler's identity:

$$e^{i\pi} + 1 = 0$$

In addition, Euler elaborated the theory of higher transcendental functions by introducing the gamma function.

Number theory

Euler pioneered the use of analytic methods to solve number problems and his work is extremely useful today in cryptography. For example, he studied the nature of prime distribution (1,2, 3, 5, 7, 11, ...) with ideas in analysis.

Euler proved Fermat's theorems, and made distinct contributions to Lagrange's four-square theorem. He invented the totient function which assigns to a positive integer n the number of positive integers less than n and coprime to n. He discovered Euler's theorem :

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

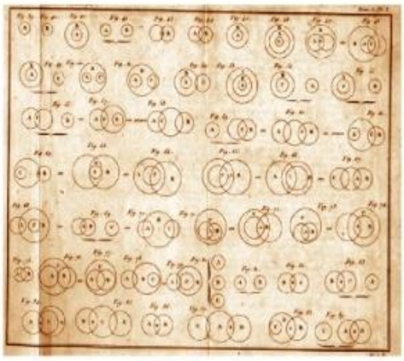
Graph theory, topology, combinatorics, operations research

In 1736 Euler solved the 7 bridges of Königsberg's problem. Königsberg has a river with two islands and seven bridges. Euler explained why it is not possible to walk on each bridge exactly once, and return to the starting point. His solution is considered the birth of graph theory and therefore of operations research.

He introduced the "Euler characteristic" notion of a space by finding a formula relating the number of edges, vertices, and faces of polyhedron. He proved Newton's identities, with an impact in Galois theroy, group therory, combinatorics, general relativity, etc.

Applied mathematics, numerical analysis

Some of Euler's greatest successes were in using analytic methods to solve real world problems. He integrated Leibniz's differential calculus with Newton's method of fluxions, and developed tools to apply calculus to physical problems. He improved the numerical approximation of integrals, inventing the Euler approximations, Euler's method and the Euler-Maclaurin formula. He also facilitated the use of differential equations.



Physics and astronomy

Euler developed the Euler-Bernoulli beam equation, which became a cornerstone of engineering. He successfully applied his analytic tools to problems in classical mechanics and celestial problems, e.g. determining the orbits of comets, calculating the parallax of the sun, or determining accurate longitude tables.

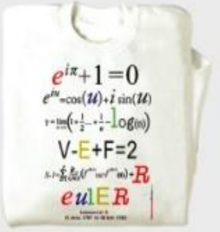
In addition, Euler made important contributions in optics. He disagreed with Newton's corpuscular theory of light, which was then the prevailing theory. His papers on optics helped ensure that the wave theory of light became the dominant mode of thought, at least until the development of the quantum theory of light.

Logic

He is also credited with using closed curves to illustrate syllogistic reasoning. These diagrams are now known as Euler diagrams, and do not need to show all possible intersections.

Euler diagrams

What is named after Euler

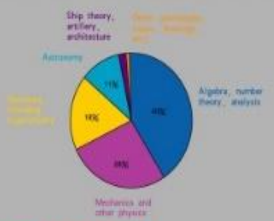


Euler's Methodus inveniendi lineas curvas - the first systematic treatise on the calculus of variations.

One of Euler's more unusual interests was the application of mathematical ideas in music.

Technical range

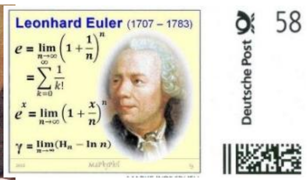
Euler's writings break down approximately as follows:

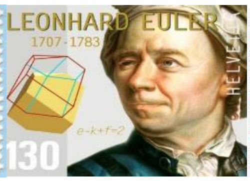


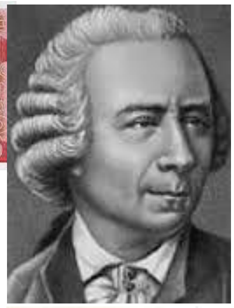
From Euler to art?

Euler's characteristic is also key in tessellation (shapes that fills the plane), together with Swiss mathematician Schläfli's symbol, to explain why there are only 17 possible "shapes" of wallpapers, or ties:

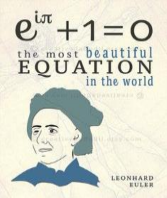
$e^{i\pi} + 1 = 0$





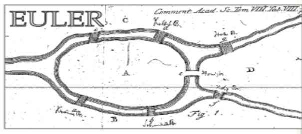


Leonhard Euler 1707–1783 one of the greatest mathematicians and the most prolific one in history (his papers were published for 50 years after his death); also physicist, astronomer, logician and engineer. Worked in all the known subjects: infinitesimal calculus, trigonometry, algebra, geometry, number theory, logic. In 1735 solved Basel Problem.



Important books: 1736 *Mechanica* is the first mechanics textbook based on differential equations. 1748 *Introductio in analysin infinitorum* (on functions), 1755 *Institutiones calculi differentialis* (on differential calculus).

Graph theory: 1736 problem known as the Seven Bridges of Königsberg, the formula V – E + F = 2



Math analysis, analytic number theory, theory of hypergeometric series, q-series, hyperbolic trigonometric functions and the analytic theory of continued fractions. He proved the infinitude of primes using the divergence of the harmonic series, and he used analytic methods to gain some understanding of the way prime numbers are distributed the sum of the reciprocals of the primes diverges.

Algebra: a new method for solving quartic equations, four-square identity

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n)\right).$$

Number theory: he proved that the relationship shown between perfect numbers and Mersenne primes earlier proved by Euclid was one-to-one (known as the Euclid–Euler theorem). He conjectured the law of quadratic reciprocity, proved Newton's identities, Fermat's little theorem, Fermat's theorem on sums of two squares. He invented the Euler function ϕ(*n*), gave numerous applications of the Bernoulli numbers, Venn diagrams, continued fractions.

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

$$e^z = \cfrac{1}{1-\cfrac{z}{1+z-\cfrac{z}{2+z-\cfrac{2z}{3+z-\cfrac{3z}{4+z-\ddots}}}}} \qquad \pi = \cfrac{4}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}}.$$

Calculus of variations including the Euler–Lagrange equation.

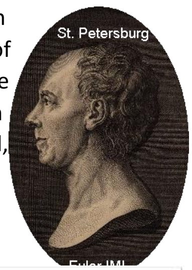
Fourier series: π/2-x/2=six x+(sin 2x)/2+(sin 3x)/3+…

Terminology and notation: f(x) in 1734, ***e*** in 1727, ***π*** in 1755, ***i*** in 1777, also **∑**, **Δy**, **Δ²y**, introduced zeta-function in 1737, gamma-function, Euler function, etc.

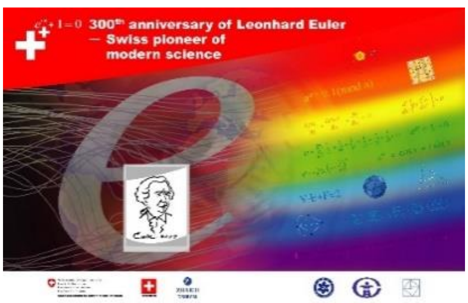
Mechanics, fluid dynamics, optics, continuum physics, lunar theory

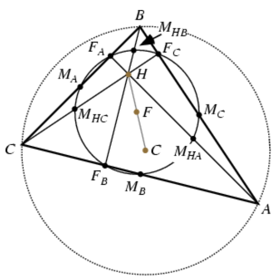
Music theory 1739 he wrote the *Tentamen novae theoriae musicae,* hoping to eventually incorporate musical theory as part of mathematics. This part of his work, however, did not receive wide attention and was once described as too mathematical for musicians and too musical for mathematicians

1720-23 Ms. Phil. at the University of Basel, student of Johann Bernoulli, 1726 thesis "*De Sono"* on the propagation of sound, failed to get a position in Basel "too young", 1727 took second place in a competition of Paris Academy of Sci. "to find the best way to place the masts on a ship" (later Euler won such annual prize twelve times), 1727 came to St.Petersburg Academy of Sci. to join Daniel Bernoulli after his brother Nicolas died (first Euler took a position in Physiology, then Math), "*Hydrodynamics of blood circulation*", 1731 Prof of Physics, 1733 after departure of Daniel, succeeded him as the head of Math, lost right eye after a fewer, 1735 became famous after solving the Basel Problem, 1737 introduced and studied zeta-function ζ(x) 1739 found $\zeta(2n)=c\pi^{2n}$, expressing c through Bernoulli numbers, 1741 moved to Berlin as a head of Math in Berlin Academy of Sci., published 380 papers in 25 years, *Letters of Euler on different Subjects in Natural Philosophy Addressed to a German Princess* (widely read in Europe and US), relation with Frederick the Great deteriorated (he called Euler "math Cyclop and made fun of him), 1748 e^{ix}=cos x+i sin x



1766 returned to Russia by invitation of Catherine the Great (huge annual salary, a pension for his wife, and the promise of high-ranking appointments for his sons); joke with Diderot: "Sir, $a+b^n/n=x$, hence God exists—reply!" 1777 found "Cauchy-Riemann equations" (d'Alembert did it in 1752 yet)

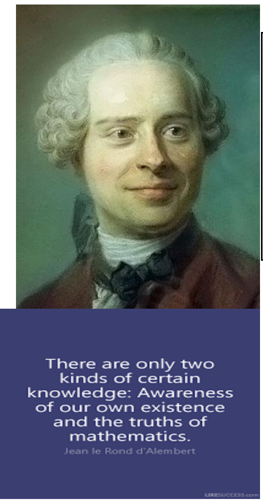




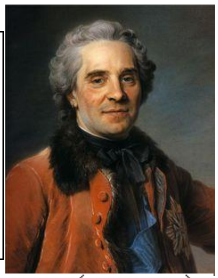
Leonhard Euler

Lecture 8. Math of 18th century, Part 2

1702 David Gregory's popular account of Newton's theories.
1707 Newton's *Arithmetica universalis* (*General Arithmetic*) with his results in algebra.
1707 De Moivre representation of complex numbers in form $r(\cos x + i \sin x)$.
1713 Jacob Bernoulli's *Ars conjectandi* (*The Art of Conjecture*) on probability.
1715 Brook Taylor *Direct and Indirect Methods of Incrementation* on calculus.
1718 Jacob Bernoulli's work on the calculus of variations is published after his death.
1718 De Moivre *The Doctrine of Chances* on probability and statistics.
1719 Brook Taylor publishes *New principles of linear perspective*.
1730 De Moivre gives Stirling's formula. $\ln n! = n \ln n - n + O(\ln n)$
1731 Clairaut publishes *Recherches sur les courbes à double coubure* on spacial curves.
1733 De Moivre describes the normal distribution curve, or law of errors (Gauss did in 1820)
1733 Saccheri an early work on non-euclidean geometry, hoped to prove the parallel postulate
1734 Berkeley *The analyst: or a discourse addressed to an infidel mathematician* "*although the calculus gave true results its foundations were no more secure than those of religion*".
1737 Simpson *Treatise on Fluxions* a textbook with infinite series and integrals of functions.
1739 D'Alembert publishes *Mémoire sur le calcul intégral* (*Memoir on Integral Calculus*).
1740 Simpson a probability treatise based on the work of de Moivre.
1740 Maclaurin is awarded the Grand Prix for the gravitational theory explaining the tides
1742 Maclaurin *Treatise on Fluxions* the first systematic exposition of Newton's methods written in reply to Berkeley's attack on the calculus for its lack of rigorous foundations.
1742 Goldbach conjectures, in a letter to Euler, that every even number ≥ 4 can be written as the sum of two primes. It is not yet known whether Goldbach's conjecture is true.
1743 D'Alembert *Treatise on Dynamics*: the principle that *the internal actions and reactions of a system of rigid bodies in motion are in equilibrium*. In 1744 (*Treatise on Equilibrium and on Movement of Fluids*) he applies his principle to the equilibrium and motion of fluids.
1746 D'Alembert's theory of complex numbers, attempted to prove the fundamental theorem of algebra
1747 D'Alembert uses partial differential equations to study the winds in *Reflection on the General Cause of Winds* which receives the prize of the Prussian Academy.
1748 Euler's *Analysis Infinitorum* (*Analysis of the Infinite*) introduced math analysis as the study of functions (rather than on geometric curves, as had been done previously). $e^{\pi i} = -1$
~1750 D'Alembert in "three-body problem" applies calculus to celestial mechanics. (Euler, Lagrange and Laplace also work on that problem.)
1750 Cramer a classification of curves, "Cramer's rule" is given.
1751 Euler theory of logarithms of complex numbers.
1752 D'Alembert: the Cauchy-Riemann equations while investigating hydrodynamics.
1752 Euler theorem $V - E + F = 2$ for polyhedra.
1753 Simson in the Fibonacci sequence F_n the ratio F_n/F_{n-1} approaches the golden ratio.
1754 Lagrange analysis of tautochrone lead to the new subject of the calculus of variations.
1755 Euler In *Institutiones calculi differentialis* introduced the calculus of finite differences. 1757 Lagrange establishes a math society in Italy that become the Turin Academy of Sciences.
1758 "Halley's comet" on 25 December confirms Halley's predictions 15 years after his death.
1759 Aepinus *An Attempt at a Theory of Electricity and Magnetism*



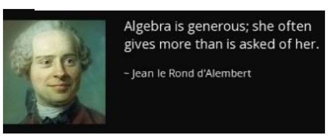
Jean Baptiste D'Alembert 1717-1783
wave equation, fluid mechanics, rival of Clairaut on dynamics, essay on winds, music theory, attempt to prove the Fundamemntal Theorem of Algebra; 1772 perpetual secretary of Royal Math Soc; 1752 Cauchi-Riemann equations

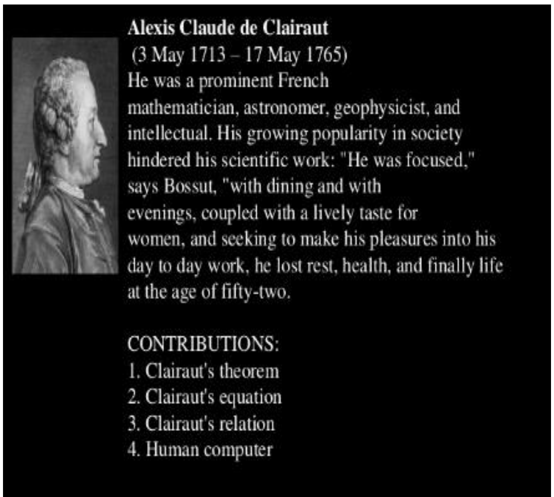


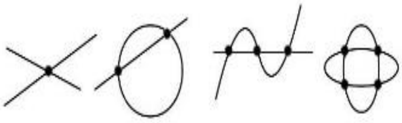
D'Alembert's principle

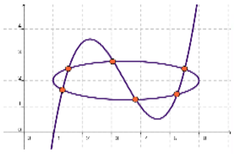
D'Alembert's principle, also known as the Lagrange–d'Alembert principle, is a statement of the fundamental traditional laws of motion.
The principle that the resultant of the external forces F and the kinetic reaction acting on a body equals zero.
D'Alembert's principle in mechanics, principle permitting the reduction of a problem in dynamics to one in statics.

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)$$





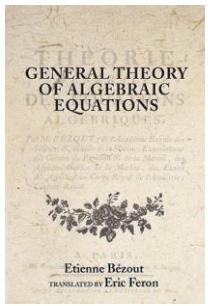




Lemma 1.1: Bézout's identity

Let a and b be nonzero integers and let d be their greatest common divisor. Then there exist integers x and y such that:

$$ax + by = d$$



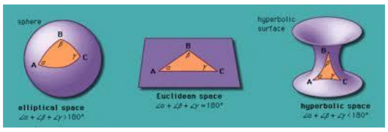
Étienne Bézout (1730-1783) 1779 *Théorie générale des équations algébriques*, theory of elimination and symmetric functions of the roots of an equation; since 1764 used determinants but did not treat the general theory. Bezout theorem (on the intersections), Little Bezout theorem (polynomial remainder theorem). Bezout identity (Lemma): the greatest common divisor is a combination $ax + by = d$ (proved for polynomials).

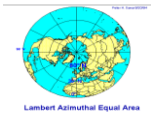
$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \ddots}}}}}$$

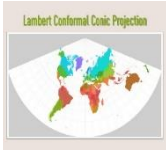
Johann Heinrich Lambert (1728-1777) a Swiss polymath credited with the proof that π is irrational (the proof uses generalized continued fraction for tan x). Non-Euclidean (hyperbolic) geometry: introduced hyperbolic functions, Lambert quadrilateral, for hyperbolic triangles showed that the sum of angles is less than π. Theory of map projections, the first to discuss the properties of conformality and equal area preservation and to point out that they were mutually exclusive; invented seven new map projections. He invented the first practical hygrometer. In 1760, he published a book on photometry, the *Photometria*.

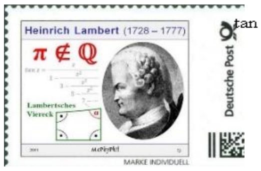
$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \ddots}}}$$

$$\tan x = \cfrac{1}{\cfrac{1}{x} - \cfrac{1}{\cfrac{3}{x} - \cfrac{1}{\cfrac{5}{x} - \cfrac{1}{7 - \dots}}}}$$



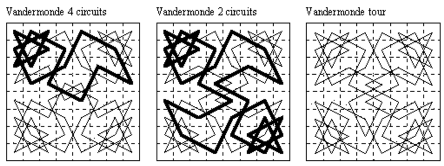






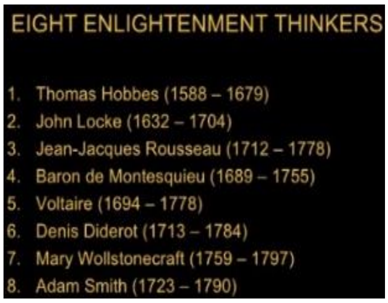
Alexandre-Théophile Vandermonde (1735-1796) math (also chemist and musician), worked with Bézout, Monge, Lavoisier; on foundations of determinant theory, on the knight tours, combinatorics, symmetric polynomials.

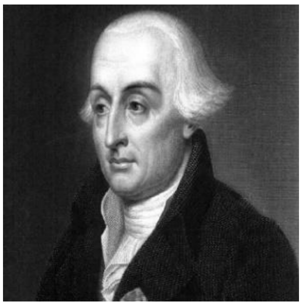
$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix}$$

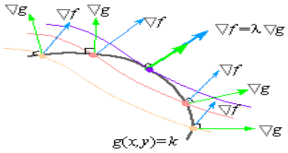


1761 Lambert proves that π is irrational. He publishes it with more general results in 1768.
1763 Monge begins the study of descriptive geometry.
1764 Bayes publishes *An Essay Towards Solving a Problem in the Doctrine of Chances* which gives Bayes theory of probability. The work contains the important "Bayes' theorem".
1765 Euler: *Theory of the Motions of Rigid Bodies* the foundation of analytical mechanics.
1766 Lambert: *Theorie der Parallellinien*, a study of the parallel postulate. In 1767 D'Alembert calls the failure to prove the parallel postulate "scandal of elementary geometry".
1769 Euler publishes the first volume of his three volume work *Dioptics*.
1769 Euler makes Euler's Conjecture: impossibility that the sum of three fourth powers is a fourth power, the sum of four fifth powers is a fifth power, and similarly for higher powers.
1770 Lagrange proves that any integer can be written as the sum of four squares. In *Réflexions sur la résolution algébrique des équations* he explains why equations of degrees ≤4 can be solved in radicals; his studying of permutations of the roots leads to group theory.
1770 Euler publishes his textbook *Algebra*.
1771 Lagrange proves Wilson's theorem (first stated without proof by Waring) that n is prime if and only if $(n - 1)! + 1$ is divisible by n.
1777 Euler used symbol i for √-1 in a manuscript which will not appear in print until 1794.
1779 Bézout *Théorie générale des équation algébraiques* including "Bézout's theorem".
1780 Lagrange wins the Grand Prix of the Paris Acad. Sci. for his work on perturbations of the orbits of comets by the planets.
1784 Legendre "Legendre polynomials" in his work *Recherches sur la figure des planètes* on celestial mechanics.
1785 Condorcet publishes *Essay on the Application of the Analysis to the Probability of Majority Decisions*. It is very important in the study of probability in the social sciences.

Cultural context: Age of Enlightment (Age of Reason): 18th century, landmarks 1715 (death of Louis 14th) – 1789 (Revolution) Philosophical movement dominated in Europe in the 18th century: ideals of reason as the primary source of authority and legitimacy, also individual liberty, progress, religious tolerance, fraternity, constitutional government and against abuses of the church and state. The most influential publication of the Enlightenment was the *Encyclopédie*, compiled by Denis Diderot and (until 1759) by Jean le Rond d'Alembert and a team of 150 scientists and philosophers. It was published between 1751 and 1772 in thirty-five volumes, and spread the ideas of the Enlightenment across Europe and beyond.







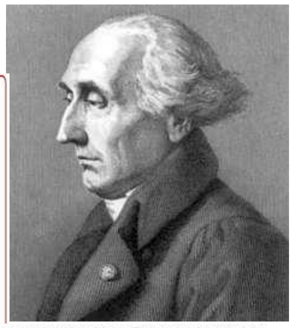
$$\nabla f(x,y,z)=\lambda \nabla g(x,y,z)+\mu \nabla h(x,y,z)$$
$$g(x,y,z)=c$$
$$h(x,y,z)=k$$

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$ [assuming that these extreme values exist and $\nabla g \neq \vec{0}$ on the curve $g(x, y) = k$]:

(a) Find all values of x, y, and λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y) = k$$

(b) Evaluate f at all points (x, y) that result from step (a). The largest of those values is the maximum value of f, the smallest is the minimum value of f.



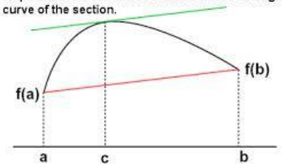
Joseph-Louis Lagrange, or Giuseppe Lodovico (or Luigi) Lagrangia 1736–1813 analysis, number theory, classical and celestial mechanics, Lagrangian mechanics (replaced Newtonian)

The Lagrangian

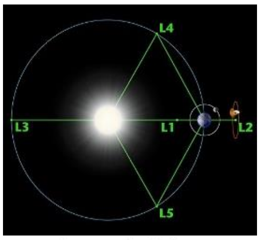
$$L = T - V$$
$$S = \int L\, dt$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

Lagrange's Mean Value Theorem states that, for any section of a continuous smooth curve, there will always be a point c at which the derivative or slope of the curve will be the same as the average slope of the curve of the section.

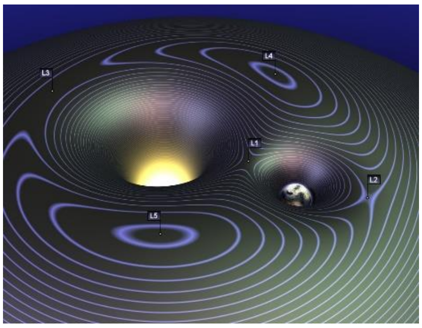


1754 the problem of tautochrone (the same as brachistocrone), discovered a method of maximizing and minimizing functionals in a way similar to finding extrema of functions, in several letters to Euler in 1754-56 described his results (the calculus of variations, the Euler–Lagrange equations); method of variation of parameters in dif. Equations, the method of Lagrange multipliers



Lagrangian Points

1766 on the proposal of Euler and d'Alembert, Lagrange succeeded Euler as the director of math at the Prussian Acad. of Sci. in Berlin



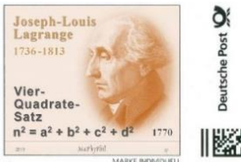
Works in Celestial mechanics: 1764-66 the libration of the Moon, explained why the same face is turned to the earth; the three-body problem for the Earth, Sun and Moon; the movement of Jupiter's satellites; 1772-76 stability of Solar system, Lagrangian points.

- **Lagrange points, L-points**, or **libration points**) are the five positions in an orbital configuration where a small object affected only by gravity can theoretically be part of a constant-shape pattern with two larger objects (such as a satellite with respect to the Earth and Moon).
- The Lagrange points mark positions where the combined gravitational pull of the two large masses provides precisely the centripetal force required to orbit with them.

1788 Analytical mechanics (*Mécanique analytique*) the most comprehensive treatment of classical mechanics since Newton, formed a basis for math physics in the 19th century

Works in Algebra: 1769-70 a tract on the Theory of Elimination, 1770 every positive integer is the sum of four squares (it was stated by Bachet without justification); 1771 Wilson's theorem:for any n>1, n is a prime if and only if n|(n − 1)! + 1; 1770-71 solving an algebraic equation of any degree via the *Lagrange resolvents*. 1773 a functional determinant of order 3 (a special case of a Jacobian); the volume of a tetrahedron via the determinant; 1773-77 proved several results enunciated by Fermat; 1775 Recherches d'Arithmétique representations of integers by quadratic forms and by more general algebraic forms: Pell's equation $x^2 - ny^2 = 1$ has a nontrivial solution in the integers for any non-square natural number n; *Theorie des fonctions analytiques*: Lagrange's theorem the order of a subgroup H of a group G must divide the order of G;

- Euler and Lagrange: Any factor of a Fermat number $2^{2^m}+1$ must be of the form $k \cdot 2^n + 1$ where $n \geq m + 2$.
- Early researchers noted that some k values gave sequences of $k \cdot 2^n + 1$ that were rich in primes, other k values gave sequences very sparse in primes.



Lagrange Polynomial:

$$p(x) = L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3 + \ldots L_N(x)y_N$$

$$L_k(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_N)}{(x_k-x_1)(x_k-x_2)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_N)}$$

Numerator of $L_k(x)$ is product of all $(x - x_i)$ except for $(x - x_k)$
Denominator of $L_k(x)$ is product of all $(x_k - x_i)$ except for $(x_k - x_k)$

Interpolation and finite differences 1783-93 via Taylor series; the theory of continued fractions, Lagrange polynomials

1787 moved from Berlin to Paris 1794 the first professor of analysis at the École Polytechnique, 1795 math Chair at Ecole Normale (after 4 months was closed), 1799 founding member of the Bureau des Longitudes and Senator, under Napoleon: Legion of Honour, 1808 Count of the Empire, buried in the Pantheon.

Prove Lagrange's Identity:
$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$

Taylor's Theorem with Remainder

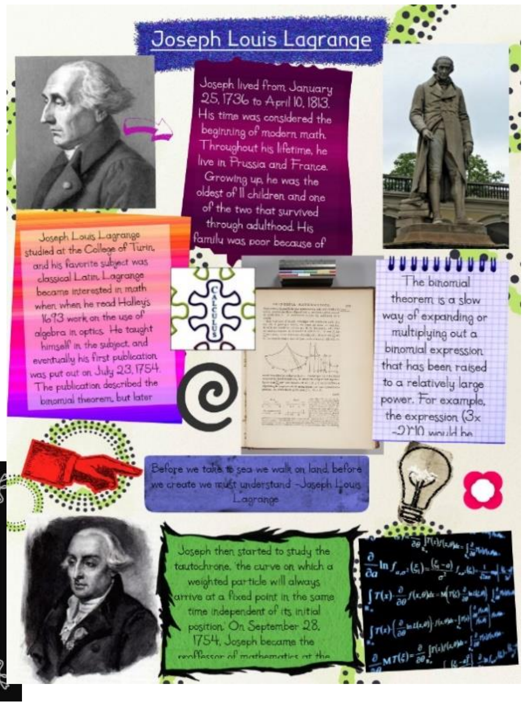
If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Lagrange Form of the Remainder

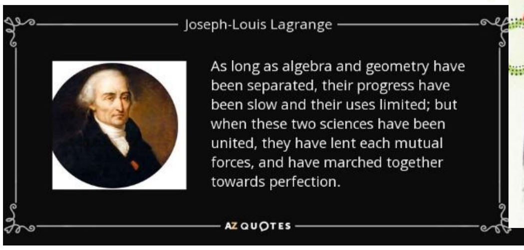
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

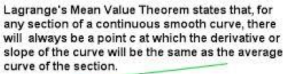
Remainder after partial sum S_n where c is between a and x.

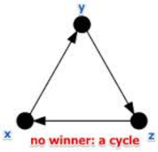


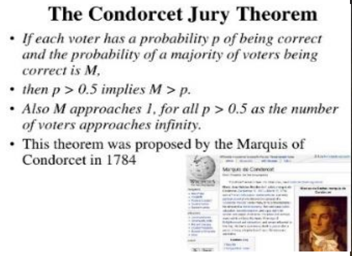
Joseph Louis Lagrange (1736-1813)

- Born in Turin at that time the capitol of Sardinia-Piemont as Giuseppe Lodovico Lagrangia
- 1774 Started correspondence with Euler and in 1775 sent him his results on the tautochrone containing his method of maxima and minima.
- 1755 was appointed professor of mathematics at the Royal Artillery School in Turin.
- 1756 he sent Euler results that he had obtained on applying the calculus of variations to mechanics.
- 1766 after having refused arrangements made by d'Alembert once and once before that Euler he accepts a post at the Berlin Academy of Science and became the successor of Euler as Director of Mathematics.
- He won prizes of the Académie des Sciences of Paris. He shared the 1772 prize on the three body problem with Euler, won the prize for 1774, another one on the motion of the moon, and he won the 1780 prize on perturbations of the orbits of comets by the planets.
- In 1770 he also presented his important work *Réflexions sur la résolution algébrique des équations* which made a fundamental investigation of why equations of degrees up to 4 could be solved by radicals
- 1787 he left Berlin to become a member of the Académie des Sciences in Paris, where he remained for the rest of his career, escaping the turmoil of the French Revolution and being decorated by Napoleon.





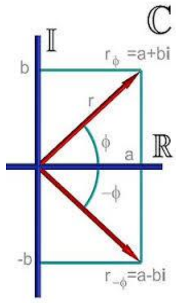



The Condorcet Jury Theorem

- *If each voter has a probability p of being correct and the probability of a majority of voters being correct is M,*
- *then p > 0.5 implies M > p.*
- *Also M approaches 1, for all p > 0.5 as the number of voters approaches infinity.*
- This theorem was proposed by the Marquis of Condorcet in 1784

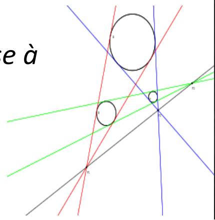
Nicolas de Condorcet, Marquis (1743–1794) philosopher,
Math: calculus and probability applied to political science
Essay on the Application of Analysis to the Probability of Majority Decisions, Condorcet Paradox, Condorcet method and Condorcet Jury Theorem (concerning voting)

Caspar Wessel (1745-1818, Copenhagen) mathematician and cartographer. In 1799 Wessel gave the geometrical interpretation of complex numbers as points in the complex plane. Since his work was in Danish in a local journal, it went unnoticed for nearly a century; the same results were independently rediscovered by Argand in 1806 and Gauss in 1831.

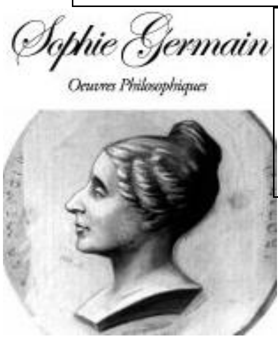


Jean-Robert Argand (1768-1822) an amateur mathematician, in 1806, while managing a book-store in Paris, he gave a geometrical interpretation of complex numbers known as the Argand diagram; he is also known for the first rigorous proof of the Fundamental Theorem of Algebra.

Gaspard Monge (1746-1818) developed analytic geometry (point-slope equation) the inventor of descriptive geometry (the mathematical basis of drawing), and the father of differential geometry, because of work *Application de l'analyse à la géométrie* where he introduced the concept of lines of curvature of a surface in 3-dimensional space. During the French Revolution he served as the Minister of the Marine, and was involved in the reform of the French educational system, helping to found the École Polytechnique.

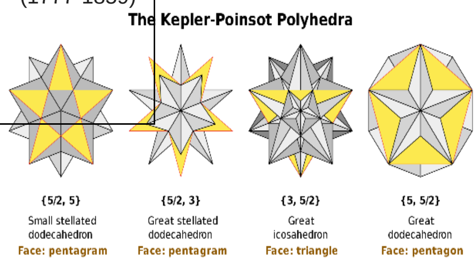


1785 Legendre states the law of quadratic reciprocity but his proof is incorrect.
1785 Lagrange begins work on elliptic functions and elliptic integrals.
1788 Lagrange *Analytical Mechanics* transforms mechanics into a branch of math. analysis.
1794 Legendre *Eléments de géométrie*, a leading textbook for 100 years that replaced Euclid's *Elements*.
1796 Laplace nebular hypothesis in *Exposition du systeme du monde* which views the solar system's origin from the contracting and cooling of a large, flattened, and slowly rotating cloud of incandescent gas.
1796 Gauss gives the first correct proof of the law of quadratic reciprocity
1797 Lagrange *Theory of Analytical Functions* the theory of functions of a real variable, notation $^{dy}/_{dx}$
1797 Wessel vector representation of complex numbers, published in Danish in 1799.
1797 Mascheroni *Geometria del compasso* all Euclidean constructions can be made with compasses alone.
1797 Lazare Carnot treats infinitely small and infinity as limits.
1799 Gauss proves the fundamental theorem of algebra correcting earlier proofs, like d'Alembert's in 1746.
1799 Laplace *Celestial Mechanics*, vol.1: examines the stability of the Solar System.
1799 Monge publishes *Géométrie descriptive:* orthographic projection
1799 Ruffini alg. equations of degree ≥5 cannot be solved in radicals. It was largely ignored as were the further proofs he would publish in 1803, 1808 and 1813.
1801 Gauss *Discourses on Arithmetic* contains seven sections, the first six of which are devoted to number theory and the last to the construction of a regular 17-gon by ruler and compasses.
1801 The minor planet Ceres is discovered but then lost. Gauss computes its orbit from the few observations that had been made leading to Ceres being rediscovered in almost exactly the position predicted by Gauss.
1801 Gauss proves Fermat's conjecture that every number can be written as the sum of three triangular numbers.
1806 Argand diagram representing complex numbers geometrically in the plane.
1806 Legendre the method of least squares for best approximations to a set of observed data.
1807 Fourier *On the Propagation of Heat in Solid Bodies:* functions via trigonometric series.
1808 Sophie Germain on the Fermat's last theorem, "Germain's theorem" as it is named by Legendre.
1809 Poinsot discovers two new regular polyhedra.
1809 Gauss used the least-squares method in *Theory of the Movement of Heavenly Bodies*.



A prime number p is a **Sophie Germain prime** if $2p + 1$ is also prime, such p are related to public key cryptography. It is conjectured that there are infinitely many Sophie Germain primes, but this has not been proven. **Sophie Germain's Identity**: $x^4+4y^4 = ((x+y)^2+y^2)((x-y)^2+y^2) = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$. If follows from **Sophie Germain's theorem** that Fermat's theorem holds for degrees n<200.

Louis Poinsot (1777-1859)

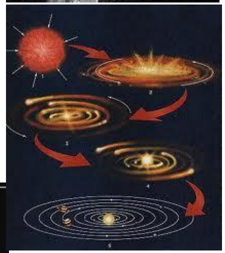


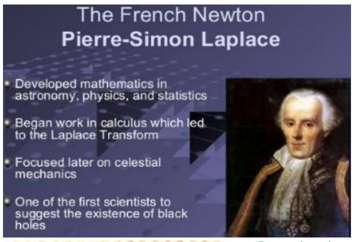
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_{0^-}^{\infty} e^{-st} f(t)\,dt \\ &= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} f'(t)\,dt \quad \text{(by parts)} \\ &= \left[-\frac{f(0^-)}{-s}\right] + \frac{1}{s}\mathcal{L}\{f'(t)\},\end{aligned}$$

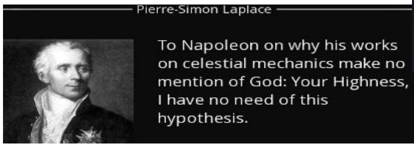
Lecture 9. Math in the end of 18th – beginning of 19th centuries

Pierre-Simon, marquis de Laplace (1749-1827) one of the greatest scientists of all time, *Newton of France*, a count of the First French Empire (1806) and a marquis (1817).



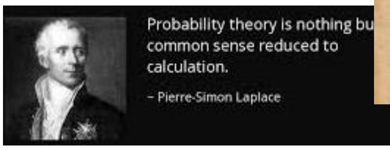


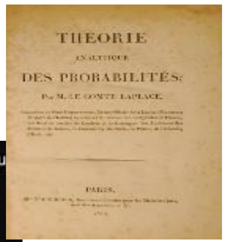
Celestial Mechanics in 5 volumes, (1799–1825) transformed the geometric study of classical mechanics to calculus, developed the nebular hypothesis of the origin of the Solar System and was the first to postulate the existence of black holes and the notion of gravitational collapse.



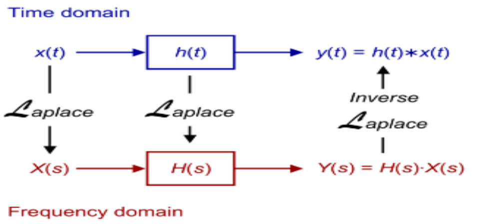
Statistics: the Bayesian interpretation of probability was developed mainly by Laplace, who developed also the characteristic function as a tool for large-sample theory and proved the first general central limit theorem. De Moivre–Laplace theorem that approximates binomial distribution with a normal distribution.

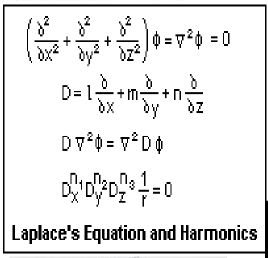
$$\Pr(A_i|B) = \Pr(A_i)\frac{\Pr(B|A_i)}{\sum_j \Pr(A_j)\Pr(B|A_j)}.$$



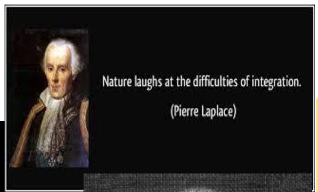


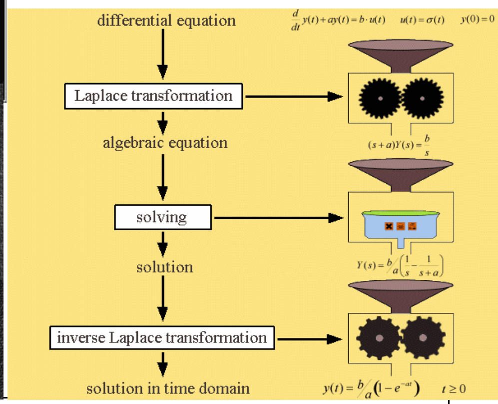
Calculus, Dif. Equations: Laplace's equation, Laplace transform appear in many branches of mathematical physics, a field that he took a leading role in forming. Laplacian differential operator, General proof of the Lagrange reversion theorem.

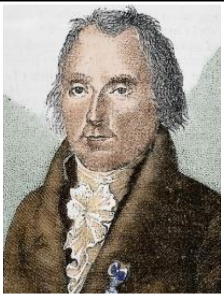


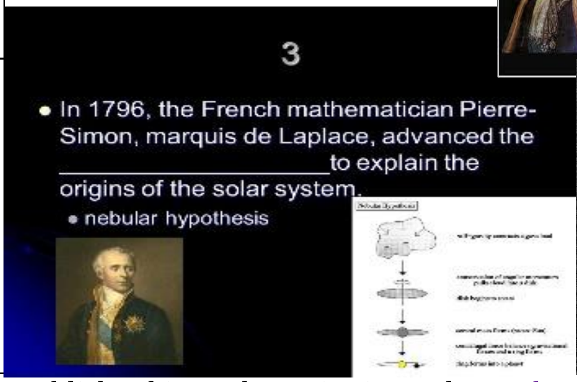


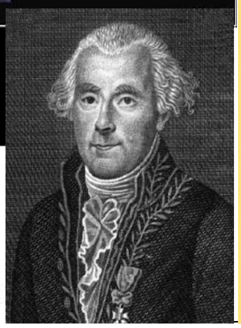
Physics: the theory of capillary action and the Young–Laplace equation.











1810-32 Gergonne publishes his mathematical journal *Annales de mathématique pures et appliquées* which became known as *Annales de Gergonne*.
1811 Poisson *Treatise on Mechanics* includes Poisson's work on the applications of math to electricity, magnetism and mechanics.
1812 Laplace *Analytical Theory of Probabilities*, vol.1: generating functions, approximations; vol.2: Laplace's definition of probability, Bayes's rule, math expectation.
1814 Argand beautiful proof (with some gaps) of the fundamental theorem of algebra.
1814 Barlow produces Barlow's Tables (factors, squares, cubes, square roots, reciprocals etc. from 1 to 10000).
1815 Peter Roget (the author of Roget's Thesaurus) invents the "log-log" slide rule.
1815 Pfaff publishes important work on what are now called "Pfaffian forms".
1817 Bessel discovers "Bessel functions", in his study of three bodies problem.
1817 Bolzano defines continuous functions without infinitesimals, the Bolzano-Weierstrass theorem.
1818 Adrain publishes *Investigation of the figure of the Earth and of the gravity in different latitudes*.
1819 Horner "Horner's method" for solving algebraic equations
1820 Brianchon publishes a statement and proof of the nine point circle theorem.

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}\left(\frac{q}{p}\right).$$

$$\left(\frac{a}{p}\right) = (a\,|\,p) \equiv \begin{cases} 0 & \text{if } p\,|\,a \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

Adrien Marie Legendre (1752–1833)

Louis Legendre (1752–1797) French politician

Important work on elliptic integrals
3 volume book published 1825–1830

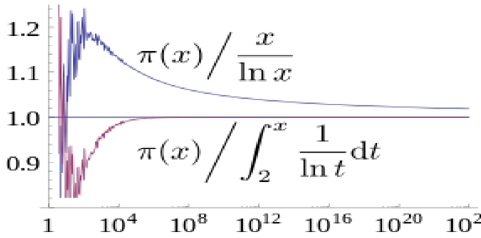
Caricature discovered in 2005

Adrien Marie Legendre in 1820

Adrien-Marie Legendre (1752-1833) Number theory, algebra, geometry, analysis, statistics

Number theory: Legendre symbol and partial proof of the quadratic reciprocity law (completed later by Gauss); 1798 Legendre conjecture on existence of primes between n^2 and $(n+1)^2$, statement of the Prime Number Theorem about the distribution of primes (proved later by Hadamard and de la Vallée-Poussin in 1896).

1. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$
2. $\left(\frac{1}{p}\right) = 1$
3. $a \equiv b \pmod{p} \;\Rightarrow\; \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$
4. $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}, \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$
5. $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$



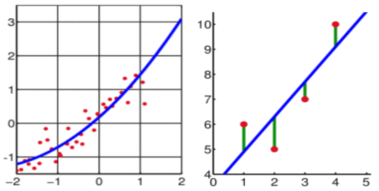
~1811 introduced the symbol Γ and name "gamma function" such that Γ(n+1) = n!. $\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt$

1830 proved the Fermat's last theorem for exponent $n = 5$ (in 1828 also by Lejeune Dirichlet).

The least squares approach to finding a "line of best fit" for a set of disparate points or observations was developed independently by Legendre and Gauss.

The method allows a summary or approximation of available data, and the prediction of other unobserved values.

It works by minimizing the sum of the squares of the errors of known data points from each side of a line.



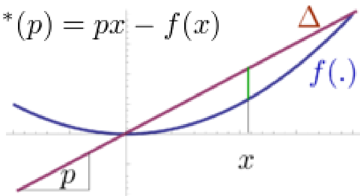
Algebra: work on roots of polynomials that inspired Galois theory.

Statistics, approximation: 1806 (appendix to a book on the paths of comets) the least squares method later developed by Gauss (used also in linear regression, signal processing, statistics, and curve fitting).

The Legendre form of an elliptic curve is given by $y^2 = x(x-1)(x-\lambda)$

Elliptic integrals: classified (later completed by Abel). An elliptic integral is an integral of the form $\int \frac{A(x)+B(x)\sqrt{S(x)}}{C(x)+D(x)\sqrt{S(x)}}\,dx.$

$$f^*(p) = px - f(x)$$

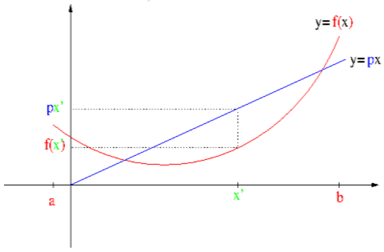


$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0.$$

Legendre transformation relates the Lagrangian to the Hamiltonian forms of mechanics, in thermodynamics it is also used to obtain the enthalpy and the Helmholtz and Gibbs energies from the internal energy. Legendre polynomials are solutions to Legendre's differential equation, which occur frequently in physics and engineering applications, *e.g.* electrostatics.

ℓ	$P_\ell(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$

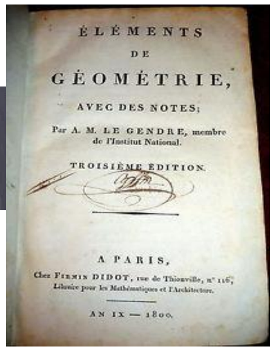


1794 *Éléments de géométrie*: greatly rearranged and simplified Euclid's *Elements*, it became the leading elementary textbook for ~100 years.

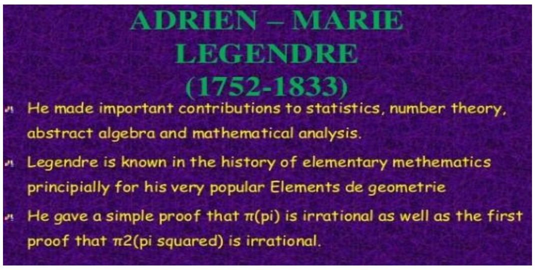
$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Adrien-Marie Legendre (1752-1833).

- 1770 defended his thesis in mathematics and physics at the Collège Mazarin
- 1775 to 1780 he taught at École Militaire
- 1782 won prize offered by the Berlin Academy for treatise on projectiles
- 1783 appointed as adjoint the Académie des Sciences, 1791 member.
- Reappointed after Napoleon
- Quarreled with Gauss over priority of reciprocity and the method of least squares.
- Worked in number theory, elliptic functions, geometry, astronomy,….
- Refused to vote for the government candidate, lost his pension and died in poverty.

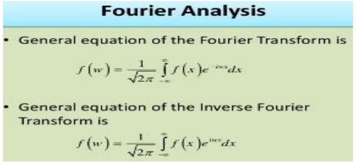


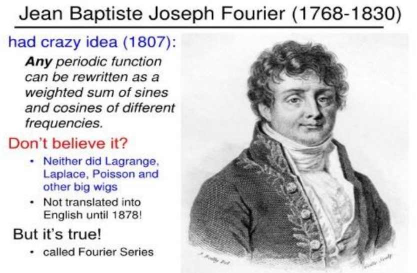
ÉLÉMENTS
DE
GÉOMÉTRIE,
AVEC DES NOTES;
Par A. M. LE GENDRE, membre de l'Institut National.
TROISIÈME ÉDITION.
A PARIS,
Chez FIRMIN DIDOT, rue de Thionville, n° 116, Libraire pour les Mathématiques et l'Architecture.
AN IX — 1800.



Paolo Ruffini (1765-1822) an Italian mathematician and philosopher. In 1799 gave an incomplete proof (Abel–Ruffini theorem) that quintic (and higher-order) equations cannot be solved by radicals, and Ruffini's rule which is a quick method for polynomial division. Ruffini also made contributions to group theory in addition to probability and quadrature of the circle. Ruffini was the first to claim the unsolvability in radicals of algebraic equations higher than quartics: this work was sent to Cauchy in 1799 which was not unckowledged. In 1801-02 it was sent 3 times to Lagrange, in 1803, 1808, 1813 new proofs were published, but nobody wanted to read. In 1821 Cauchy anknowledged correctness.

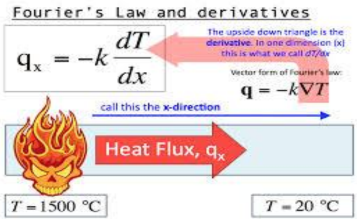
$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$

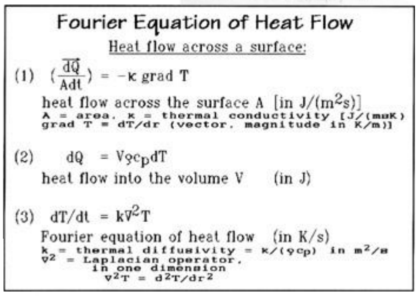




Jean-Baptiste Joseph Fourier (1768-1830) known for the Fourier series and their applications to problems of heat transfer and vibrations; Fourier transform and Fourier's law; he is credited for discovery of the greenhouse effect.

1822 *The Analytic Theory of Heat* developed Newton's law of cooling, gave heat flow law (Fourier Law)





1831 (edited by Claude-Louis Navier) Fourier's theorem on the position of the roots of an algebraic equation; the concept of dimensional homogeneity in equations (dimensional analysis); PDE for conductive diffusion of heat.

THE LAWS OF HEAT TRANSFER.

1.Fourier's law of heat conduction.

The rate of heat flow is proportional to the product of the area of flow A, and the temperature gradient (-dt/dx),the constant of proportionality being the thermal conductivity k, which is a property of material.

Unit is j/s or W

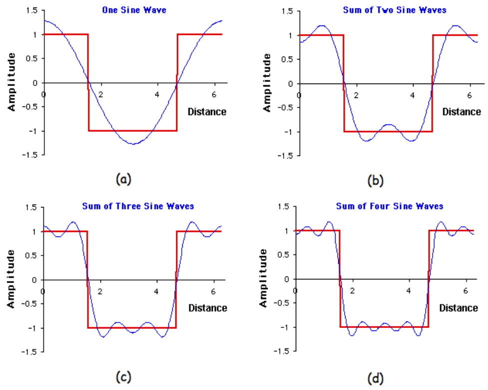
Q_x=-kA(dt/dx)

FOURIER SERIES

- Periodic signal expressed in terms of sines and cosines.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt, \quad n \geq 0 \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt, \quad n \geq 1$$



$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = -\frac{\rho_e}{\epsilon\epsilon_0}$$

Siméon Denis Poisson (1781-1840) geometer; a new branch of mathematical physics: the theory of electricity and magnetism,

1813 in *Bulletin de la société philomatique* Poisson's equation

Poisson's Equation: $\nabla^2 V = -\frac{\rho_V}{\epsilon}$

Laplace's Equation: $\nabla^2 V = 0$

Poisson distribution in probability theory

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

If, $n \to \infty, p \to 0$ so that $np \to \lambda$ then

$$\frac{n!}{(n-k)!k!}p^k(1-p)^{n-k} \to e^{-\lambda}\frac{\lambda^k}{k!}.$$

1831 Poisson derived the Navier–Stokes equations independently of Claude-Louis Navier

$$\{F, G\} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right)$$

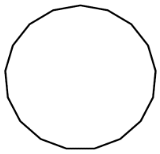
Johann Friedrich Pfaff (1765-1825) a precursor of the German school of mathematics, teacher of Carl Friedrich Gauss.

$$\mathrm{pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc.$$

Lecture10. Math in the beginning of 19th century

Johann Carl Friedrich Gauss (1777–1855) *Princeps mathematicorum*
number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, mechanics, electrostatics, astronomy, matrix theory, and optics.

1796 a regular polygon can be constructed by compass and straightedge if and only if the number of sides is the product of distinct Fermat primes and a power of 2. Any number is a sum of 3 triangular numbers: recorded in his diary "ΕΥΡΗΚΑ! num = Δ + Δ + Δ".

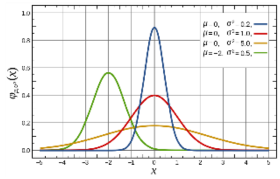


The heptadecagon (or 17-gon) is a regular 17-sided polygon which Gauss showed could be constructed geometrically.

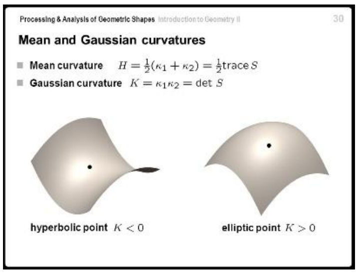
1798 *Disquisitiones Arithmeticae* (published in 1801) consolidating number theory as a discipline and has shaped the field to the present day; the quadratic reciprocity law.

1801 a position for Ceres (that was "lost" by astronomers) is predicted very accurately; 1807 was appointed Professor of Astronomy and Director of the astronomical observatory in Göttingen, a post he held for the remainder of his life.

1809 a theory of the motion of planetoids disturbed by large planets, published in *Theory of motion of the celestial bodies moving in conic sections around the Sun* that still remains a cornerstone of astronomical computation; Gaussian gravitational constant, the method of least squares, proved under the assumption of normally distributed errors, Gauss–Markov theorem. The method had been described earlier by Adrien-Marie Legendre in 1805, but Gauss claimed that he had been using it since 1795.



1818 geodesic survey of the Kingdom of Hanover, invented the heliotrope, an instrument that uses a mirror to reflect sunlight over great distances, to measure positions.
differential geometry and topology, fields of mathematics dealing with curves and surfaces. Among other things he came up with the notion of Gaussian curvature.
1828 Theorema Egregium (*remarkable theorem*): the curvature of a surface is an intrinsic property (independent of the embedding of a surface into 3-space).



Bolyai's son, János Bolyai, discovered non-Euclidean geometry in 1829; published in 1832. After seeing it, Gauss wrote to Farkas Bolyai: *"To praise it would amount to praising myself. For the entire content of the work ... coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years."* Gauss was aware of non-Euclidean geometry before it was published by Bolyai, but that he refused to publish any of it because of his fear of controversy.

1831 collaboration with the physicist Wilhelm Weber, study of magnetism (including finding a representation for the unit of magnetism in terms of mass, charge, and time); discovery of Kirchhoff's circuit laws in electricity. They constructed the first electromechanical telegraph in 1833, which connected the observatory with the institute for physics in Göttingen. Gauss ordered a magnetic observatory to be built in the garden of the observatory, and with Weber founded the "Magnetischer Verein" (*magnetic club* in German), which supported measurements of Earth's magnetic field in many regions of the world. He developed a method of measuring the horizontal intensity of the magnetic field which was in use well into the second half of the 20th century, and worked out the mathematical theory for separating the inner and outer (magnetospheric) sources of Earth's magnetic field.

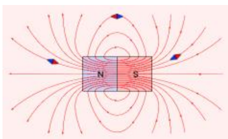
1840 *Dioptrische Untersuchungen*, images under a paraxial approximation (Gaussian optics): an optical system can be characterized by its cardinal points; Gaussian lens formula

Gauss was known to dislike teaching and attended only a single scientific conference, which was in Berlin in 1828. However, several of his students became influential mathematicians, among them Richard Dedekind, Bernhard Riemann, and Friedrich Bessel.

Gauss' Law For Magnetism

$\nabla \cdot \mathbf{B} = 0$ (Magnetic Charge Does Not Exist)

$\nabla \cdot \mathbf{H} = 0$ (also true since $\mathbf{B} = \mu\mathbf{H}$)



Gauss' Law

Gauss' Law for Electricity

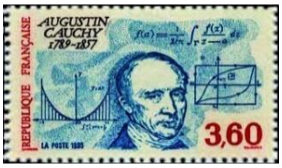
$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{inside}}{\varepsilon_0}$

Gauss' Law for Magnetism

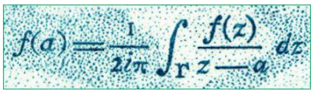
$\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0$

(no magnetic monopoles, no q_m)


The History of
Johann Friedrich Carl Gauss
1777 - 1855
I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect.
Found Bell Curve,
Graphical Presentation of Gaussian normal distribution in probability
Died, February 23 Gottingen, Germany
1855
Life stands before me like an eternal spring with new and brilliant clothes.
Mathematicians stand on each other's shoulders.
Born April 30th 1777
Brunswick, Germany
Gaus & Wilhelm Weber Invented the first Electric Telegraph.
Contribution
Findings
Geometry, Statistics, Number Theory, Planetary Astronomy
1801
Disquisitiones Arithmeticae published
1800
Create Method of Least Squares
Mapping the state of Hannover, indispensable tool for analyzing Data
The theory of functions, potential theory, optics and geophysics
1801
Found a way to construct the regular with Seventeen Sides
1797
Pioneer in Non-Euclidian Geometry
1799
Create Basic Algebra Theorem
Resources



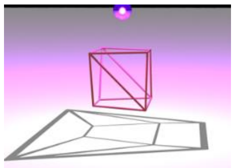
- French, 1789–1857
- Royalist and Catholic
- made contributions in geometry, calculus, complex analysis, number theory
- created the definition of limit we use today but

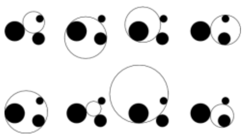


Baron **Augustin-Louis Cauchy** (1789-1857) a pioneer of analysis, stated and proved theorems of calculus rigorously, rejecting the heuristic principles; founded complex analysis studied of permutation groups in algebra.

$$f(a) = \frac{1}{2i\pi}\int_{\Gamma}\frac{f(z)}{z-a}\,dz$$

"*More concepts and theorems have been named for Cauchy than for any other mathematician* (in elasticity there are sixteen concepts and theorems named for Cauchy)." Cauchy was a prolific writer: ~800 research articles and 5 complete textbooks.





Early works: 1805 solved Problem of Apollonius (find a circle touching three given circles); 1811 generalized Euler's formula from polyhedra to planar graphs; 1813: the Fermat polygonal number theorem.

Augustin-Louis Cauchy

Calculus transformed into Analysis: 1821 ***Cours d'Analyse*** "*the man who taught rigorous analysis to all of Europe.*" Definition of continuity: *The function f(x) is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself.* Infinitesimals were defined in terms of a sequence tending to zero. The notion of convergence, the theory of series, the theory of functions, differential equations.

COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

1re PARTIE. ANALYSE ALGÉBRIQUE.

DE L'IMPRIMERIE ROYALE.

1821

$$\det(AB) = \sum_{S\in\binom{[n]}{m}} \det(A_{[m],S})\det(B_{S,[m]}).$$

$$|\langle x,y\rangle|^2 \le \langle x,x\rangle \cdot \langle y,y\rangle$$

Abstract Algebra: the theory of permutation groups and substitutions, cycle decomposition theorem, determinants; Cauchy-Binet formula, Cauchy-Schwarz inequality.

Math Physics: 1816 memoir on wave propagation (Grand Prix of the French Acad of Sci), theory of light (Fresnel's wave theory), dispersion and polarization of light; mechanics, the principle of the continuity of matter; equilibrium of rods and elastic membranes, waves in elastic media; in elasticity the theory of stress (3 × 3 symmetric matrix known as the Cauchy stress tensor)

$$f(z) = \phi(z) + \frac{B_1}{z-a} + \frac{B_2}{(z-a)^2} + \cdots + \frac{B_n}{(z-a)^n}, \quad B_i, z, a \in \mathbb{C},$$

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz, \qquad \operatorname{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)f(z), \qquad \frac{1}{2\pi i}\oint_C f(z)dz = \sum_{k=1}^{n}\operatorname{Res}_{z=a_k} f(z),$$

Complex function theory: 1814-1825 defined complex numbers as pairs of real numbers *Cauchy's integral theorem*, 1826 a residue of a function, poles, 1831 Cauchy's integral formula, residue theorem. Only in the 1840s the theory started to get response, with Pierre-Alphonse Laurent (1843 Laurent series).

$$\oint_C f(z)dz = 0,$$

Cauchy-Riemann Equation

$$f(z) \text{ is holomorphic} \Leftrightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{i\partial y}$$

$$\Updownarrow$$

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(-i\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial y}\right) = 0$$

Augustin Cauchy

- August 21, 1789 – May 23, 1857
- 1810 - Graduated in civil engineering and went to work as a junior engineer where Napoleon planned to build a naval base
- 1812 – (age 23) Lost interest in engineering, being more attracted to abstract mathematics
- Cauchy had many major accomplishments in both mathematics and science in areas such as complex functions, group theory, astronomy, hydrodynamics, and optics
- Cauchy made 789 contributions to scientific journals
- One of his most significant accomplishments involved determining when an infinite series will converge on a solution
- In wave theory, he defined an empirical relationship between the refractive index and wavelength of light for transparent materials -- Cauchy's Dispersion Equation

Cauchy integral theorem

$$\oint_\gamma f(z)dz = 0$$

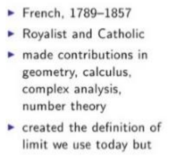
Cauchy integral formula

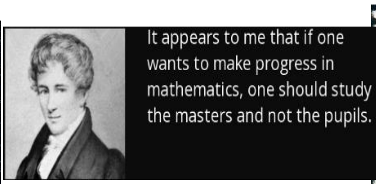
$$f(a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}dz$$

Cauchy differentiation formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}dz$$

Augustin Louis Cauchy





Niels Henrik Abel (1802–1829) proved impossibility of solving the general quintic equation in radicals; worked on the elliptic integrals and elliptic functions, discovered Abelian functions. Abel was unrecognized during his lifetime, lived in poverty and died at age 26.

1824 *Memoir on algebraic equations, in which the impossibility of solving the general equation of the fifth degree is proven*; only six pages (to save money on printing). 1826 more detailed proof in the first volume of *Crelle's Journal*.

1825 received a grant for research travel to Göttingen to visit Gauss and Paris, but first he went to Berlin (4 months) and met August Leopold Crelle, who was about to publish *Journal für die reine und angewandte Mathematik* and Abel contributed 7 papers to it in its first year. In Leipzig and Freiberg Abel did research in the theory of functions: elliptic (double periodicity), hyperelliptic, and a new class of abelian functions. From Freiberg he went to Dresden, Prague, Vienna, Trieste, Venice, Verona, Bolzano, Innsbruck, Luzern and Basel and finally Paris. 1826 submitted a memoir on on addition of algebraic differentials to French Acad. of Sci. It was reviewed by Cauchy, but was put aside and forgotten until his death. Notes: "Paris mathematicians are interested only in astronomy, heat theory, optic and elasticity. Cauchy is the only pure mathematician, although he is absolutely insane and there is nothing to do with it." 1827 return to Berlin, declined to work as an editor of Crelle's Journal and was back in Norway. His tour was viewed as a failure: not visited Gauss in Göttingen, not published anything in Paris, so, his scholarship was not renewed and he started tutoring while sending most of his work to Crelle's Journal. While in Paris, Abel contracted tuberculosis and died just two days before a letter from Crelle, informing that he found for him a position of Prof. at the Univ. of Berlin.


August Leopold Crelle 1780-1855

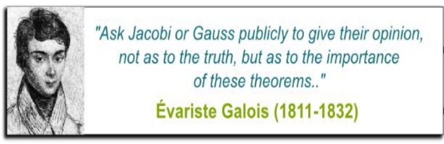
Évariste Galois (1811-1832) found necessary and sufficient condition for a polynomial to be solvable by radicals. His work laid the foundations for Galois theory and group theory, two major branches of abstract algebra. He died at age 20 from wounds suffered in a duel.

1829 submitted two papers on the polynomial equations to the Academy of Sciences, but Cauchy who refereed these papers, did not accept them for publication, although recognized the their importance and suggested to combine the two papers into one and to submit it for the Academy's Grand Prize. In 1830, Galois submitted his work to the Academy's secretary Fourier, for the Grand Prix, but Fourier died soon and the manuscript was lost. Galois published three other papers that year, one of which laid the foundations for Galois theory, the second one was about finding the roots of equations, and from the third a notion of finite field was developed.

Galois was the first to use the word *group* in a modern sense, he developed the concept of normal subgroup. Finite fields are called now Galois field since he introduced them and understood like in modern times.

In his last letter to Chevalier and attached manuscripts, the second of three, he made basic studies of linear groups over finite fields: constructed the linear group GL(v, p) over a prime field, as well as the projective group PSL(2,p) viewed via fractional linear transforms, and observed that they were simple except if p was 2 or 3.

Evariste Galois
(1811-1832)



- 1827 developed interest for mathematics
- 1828 failed the entrance exam to the Ecole Polytechnique, but continues to work on his own
- 1829 first mathematics paper published on continued fractions in the *Annales de mathématiques*.
- 25 May and 1 June he submitted articles on the algebraic solution of equations to the Académie des Sciences. Cauchy was appointed as referee of Galois' paper.
- 1829 entered the Ecole Normale.
- 1830 learned of a posthumous article by Abel which overlapped with a part of his work and submitted a new article *On the condition that an equation be soluble by radicals* in February. The paper was sent to Fourier, the secretary of the Academy, to be considered for the Grand Prize in mathematics. Fourier died in April 1830 and Galois' paper was never subsequently found and so never considered for the prize which went to Abel and Jacobi
- 1830 He published three papers in *Bulletin de Férussac*.
- 1830 Galois was invited by Poisson to submit a third version of his memoir on an equation
- 1830 For writing a political letter Galois was expelled and he joined On 31 December 1830 the Artillery of the National Guard which was subsequently was abolished by Royal Decree since the new King Louis-Phillipe felt it was a threat to the throne.
- 1830 In and out of prison
- 1832 Galois contracted cholera during the Paris epidemic. He apparently fell in love with Stephanie-Felice du Motel, the daughter of the his physician
- 1832 Galois fought a duel with Perscheux d'Herbinville on 30 May probably about Stephaine and subsequently died in Cochin hospital on 31 May.
- 1846 Liouville published the papers of Galois in his Journal.

Last Night

- ~~1831:~~ Galois again submitted to Académie des Sciences; Poisson was the Reviewer. He did not understand the paper and rejected it.
- night of 30 May 1832: injured at the duel with Perscheux d'Herbinville over the prison's physician's daughter named Stephanie-Felice du Motel: abandoned by both Perscheux as well as his seconds. A peasant took him to a hospital, where he died at the age of 21 in 1832.