47 Comparison Tests for Improper Integrals

Sometimes it is difficult to find the exact value of an improper integral by antidifferentiation, for instance the integral \( \int_0^\infty e^{-x^2} \, dx \). However, it is still possible to determine whether an improper integral converges or diverges. The idea is to compare the integral to one whose behavior we already know, such us

- the p-integral \( \int_1^\infty \frac{1}{x^p} \, dx \) which converges for \( p > 1 \) and diverges otherwise;
- the integral \( \int_0^\infty e^{cx} \, dx \) which converges for \( c < 0 \) and diverges for \( c \geq 0 \);
- the integral \( \int_0^1 \frac{1}{x^p} \, dx \) which converges for \( p < 1 \) and diverges otherwise.

The comparison method consists of the following:

**Theorem 47.1**

Suppose that \( f \) and \( g \) are continuous and \( 0 \leq g(x) \leq f(x) \) for all \( x \geq a \). Then

(a) if \( \int_a^\infty f(x) \, dx \) is convergent, so is \( \int_a^\infty g(x) \, dx \)

(b) if \( \int_a^\infty g(x) \, dx \) is divergent, so is \( \int_a^\infty f(x) \, dx \).

This is only common sense: if the curve \( y = g(x) \) lies below the curve \( y = f(x) \), and the area of the region under the graph of \( f(x) \) is finite, then of course so is the area of the region under the graph of \( g(x) \). Similar results hold for the other types of improper integrals.

**Example 47.1**

Determine whether \( \int_1^\infty \frac{1}{\sqrt{x^3+5}} \, dx \) converges.

**Solution.**

For \( x \geq 1 \) we have that \( x^3 + 5 \geq x^3 \) so that \( \sqrt{x^3+5} \geq \sqrt{x^3} \). Thus, \( \frac{1}{\sqrt{x^3+5}} \leq \frac{1}{\sqrt{x^3}} \). Letting \( f(x) = \frac{1}{\sqrt{x^3}} \) and \( g(x) = \frac{1}{\sqrt{x^3+5}} \) then we have that \( 0 \leq g(x) \leq f(x) \). From the previous section we know that \( \int_1^\infty \frac{1}{\sqrt{x^3}} \, dx \) is convergent, a p-integral with \( p = \frac{3}{2} > 1 \). By the comparison test, \( \int_1^\infty \frac{1}{\sqrt{x^3+5}} \, dx \) is convergent.

The next question is to estimate such a convergent improper integral.

**Example 47.2**

Estimate the value of \( \int_1^\infty \frac{1}{\sqrt{x^3+5}} \, dx \) with an error of less than 0.01.
Solution.
We want to find $b$ such that

$$\left| \int_1^\infty \frac{1}{\sqrt{x^3+5}} \, dx - \int_1^b \frac{1}{\sqrt{x^3+5}} \, dx \right| < 0.01.$$ 

But

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}} \, dx = \int_1^b \frac{1}{\sqrt{x^3+5}} \, dx + \int_b^\infty \frac{1}{\sqrt{x^3+5}} \, dx.$$ 

Thus, the problem is to find $b$ such that

$$\left| \int_b^\infty \frac{1}{\sqrt{x^3+5}} \, dx \right| < 0.01.$$ 

From the example above, we have

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}} \, dx < \int_1^\infty \frac{1}{x^3} \, dx = \frac{2}{\sqrt{b}}.$$ 

So it suffices to choose $b$ such that $\frac{2}{\sqrt{b}} < 0.01$ or $b > 40,000$, say for example $b = 45000$. In this case,

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}} \, dx \approx \int_1^{45,000} \frac{1}{\sqrt{x^3+5}} \, dx = 1.69824.$$

Example 47.3
Investigate the convergence of $\int_4^\infty \frac{dx}{\ln x - 1}$.

Solution.
For $x \geq 4$ we know that $\frac{1}{\ln x - 1} < \frac{1}{\ln x} < x$. Thus, $\frac{1}{\ln x - 1} > \frac{1}{x}$. Let $g(x) = \frac{1}{x}$ and $f(x) = \frac{1}{\ln x - 1}$. Thus, $0 < g(x) \leq f(x)$. Since $\int_4^\infty \frac{1}{x} \, dx = \int_4^\infty \frac{1}{\ln x - 1} \, dx - \int_4^\infty \frac{1}{x} \, dx$ which is divergent since $\int_4^\infty \frac{1}{x} \, dx$ is divergent being a p-integral with $p = 1$.

By the comparison test $\int_4^\infty \frac{dx}{\ln x - 1}$ is divergent.

Example 47.4
Investigate the convergence of the improper integral $\int_1^\infty \frac{\sin x + 3}{\sqrt{x}} \, dx$.

Solution.
We know that $-1 \leq \sin x \leq 1$. Thus $2 \leq \sin x + 3 \leq 4$. Since $x \geq 1$, we have $\frac{2}{\sqrt{x}} \leq \frac{\sin x + 3}{\sqrt{x}} \leq \frac{4}{\sqrt{x}}$. Note that the two integrals $\int_1^\infty \frac{2}{\sqrt{x}} \, dx$ and $\int_1^\infty \frac{4}{\sqrt{x}} \, dx$
are both divergent being a multiple of a p-integral with \( p = \frac{1}{2} < 1 \). If we let \( g(x) = \frac{\sin x + 3}{\sqrt{x}} \) and \( f(x) = \frac{4}{\sqrt{x}} \) then we have no conclusion since \( \int_1^\infty g(x)\,dx \) may or may not converge and still \( \int_1^\infty g(x)\,dx \leq \int_1^\infty f(x)\,dx \). Now if we let \( g(x) = \frac{2}{\sqrt{x}} \) and \( f(x) = \frac{\sin x + 3}{\sqrt{x}} \) then by the comparison test \( \int_1^\infty \frac{\sin x + 3}{\sqrt{x}} \) is divergent since \( \int_1^\infty f(x)\,dx \geq \int_1^\infty g(x)\,dx \) and \( \int_1^\infty g(x)\,dx \) is divergent.

**Example 47.5**
Investigate the convergence of \( \int_1^\infty e^{-\frac{1}{2}x^2}\,dx \).

**Solution.**
If \( x \geq 2 \) then \( \frac{x}{2} \geq 1 \). Multiply both sides of this inequality by \( x \geq 2 \) to obtain \( \frac{1}{2}x^2 \geq x \). Now, multiply both sides of this last inequality by \(-1\) to obtain \( \frac{1}{2}x^2 \leq -x \) and therefore \( e^{-\frac{1}{2}x^2} \leq e^{-x} \) since the function \( e^x \) is an increasing function. Thus,

\[
\int_1^\infty e^{-\frac{1}{2}x^2}\,dx = \int_1^2 e^{-\frac{1}{2}x^2}\,dx + \int_2^\infty e^{-\frac{1}{2}x^2}\,dx.
\]

But \( \int_1^2 e^{-\frac{1}{2}x^2}\,dx \approx 0.34 \) and

\[
\int_2^\infty e^{-\frac{1}{2}x^2}\,dx \leq \int_2^\infty e^{-x}\,dx \leq \int_0^\infty e^{-x}\,dx
\]

so since \( \int_0^\infty e^{-x}\,dx \) is convergent, \( \int_2^\infty e^{-\frac{1}{2}x^2}\,dx \) is convergent. In conclusion, \( \int_1^\infty e^{-\frac{1}{2}x^2}\,dx \) is convergent.

Sometimes it is laborious to find a convenient function \( f(x) \) with \( g(x) \leq f(x) \), but we may know that \( g(x) \) is no larger than a constant multiple of \( f(x) \) for large enough \( x \), and this is good enough. The most powerful test of this form in the course is this version of the limit comparison test:

**Theorem 47.2**
Let \( f(x) \) and \( g(x) \) be two positive and continuous functions on \([a, \infty)\).

(a) \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \), or
(b) \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = L \), where \( L \) is a finite positive number, or
(c) \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \), then
(a) If \( \int_a^\infty g(x) \, dx \) converges, then so does \( \int_a^\infty f(x) \, dx \).
(b) \( \int_a^\infty g(x) \, dx \) converges if and only if \( \int_a^\infty f(x) \, dx \) does.
(c) If \( \int_a^\infty g(x) \, dx \) diverges, then so does \( \int_a^\infty f(x) \, dx \).

**Proof.**

(a) Suppose that \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \). Let \( \epsilon > 0 \) be given. Then there is a \( b > a \) such that \( \frac{f(x)}{g(x)} < \epsilon \) for all \( x \geq b \). Thus, \( f(x) < \epsilon g(x) \) for all \( x \geq b \). By the comparison test, if \( \int_a^\infty g(x) \, dx \) is convergent so does \( \int_a^\infty f(x) \, dx \).

(b) Now, suppose that \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = L \), where \( L \) is a finite positive constant. Let \( \epsilon < L \). Then there is a constant \( b > a \) such that for all \( x \geq b \) we have

\[
\left| \frac{f(x)}{g(x)} - L \right| < \epsilon.
\]

That is,

\[
L - \epsilon < \frac{f(x)}{g(x)} < L + \epsilon.
\]

Thus, for \( x \geq b \) we have \((L - \epsilon)g(x) < f(x) < (L + \epsilon)g(x)\). Now the result follows from the comparison test.

(c) Finally, suppose that \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \). Then there is \( b > a \) such that \( \frac{f(x)}{g(x)} \geq 1 \) for all \( x \geq b \). That is, \( g(x) \leq f(x) \) for all \( x \geq b \). Therefore, if \( \int_a^\infty f(x) \, dx \) converges, then \( \int_a^\infty g(x) \, dx \) converges.

**Remark 47.1**

The Comparison Test and Limit Comparison Test also apply, modified as appropriate, to other types of improper integrals.

**Example 47.6**

Show that the improper integral \( \int_1^\infty \frac{1}{1+x^2} \, dx \) is convergent.

**Solution.**

Since the integral \( \int_1^\infty \frac{dx}{x^2} \) is convergent (\( p \)-integral with \( p = 2 > 1 \)) and since \( \lim_{x \to \infty} \frac{x^2}{x^2} \, dx = \lim_{x \to \infty} \frac{x^2}{x^2+1} = 1 \), by the limit comparison test (Theorem 47.2 (b)) we have \( \int_1^\infty \frac{dx}{x^2+1} \) is also convergent.