Arkansas Tech University
MATH 2924: Calculus II
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47 Comparison Tests for Improper Integrals

Sometimes it is difficult to find the exact value of an improper integral by antidifferentiation, for instance the integral $\int_0^\infty e^{-x^2}dx$. However, it is still possible to determine whether an improper integral converges or diverges. The idea is to compare the integral to one whose behavior we already know, such us

- the p-integral $\int_1^\infty \frac{1}{x^p}dx$ which converges for $p > 1$ and diverges otherwise;
- the integral $\int_0^\infty e^{cx}dx$ which converges for $c < 0$ and diverges for $c \geq 0$;
- the integral $\int_0^1 \frac{1}{x^p}dx$ which converges for $p < 1$ and diverges otherwise.

The comparison method consists of the following:

Theorem 47.1
Suppose that f and g are continuous and $0 \leq g(x) \leq f(x)$ for all $x \geq a$. Then

(a) if $\int_a^\infty f(x)dx$ is convergent, so is $\int_a^\infty g(x)dx$
(b) if $\int_a^\infty g(x)dx$ is divergent, so is $\int_a^\infty f(x)dx$.

This is only common sense: if the curve $y = g(x)$ lies below the curve $y = f(x)$, and the area of the region under the graph of $f(x)$ is finite, then of course so is the area of the region under the graph of $g(x)$. Similar results hold for the other types of improper integrals.

Example 47.1
Determine whether $\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx$ converges.

Solution.
For $x \geq 1$ we have that $x^3 + 5 \geq x^3$ so that $\sqrt{x^3+5} \geq \sqrt{x^3}$. Thus, $\frac{1}{\sqrt{x^3+5}} \leq \frac{1}{\sqrt{x^3}}$. Letting $f(x) = \frac{1}{\sqrt{x^3}}$ and $g(x) = \frac{1}{\sqrt{x^3+5}}$ then we have that $0 \leq g(x) \leq f(x)$. From the previous section we know that $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}dx$ is convergent, a p-integral with $p = \frac{3}{2} > 1$. By the comparison test, $\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx$ is convergent.■

The next question is to estimate such a convergent improper integral.

Example 47.2
Estimate the value of $\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx$ with an error of less than 0.01.

Solution.
We want to find b such that

$$\left|\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx - \int_1^b \frac{1}{\sqrt{x^3+5}}dx\right| < 0.01.$$

But

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx = \int_1^b \frac{1}{\sqrt{x^3+5}}dx + \int_b^\infty \frac{1}{\sqrt{x^3+5}}dx.$$

Thus, the problem is to find b such that

$$\left|\int_b^\infty \frac{1}{\sqrt{x^3+5}}dx\right| < 0.01.$$

From the example above, we have

$$\int_b^\infty \frac{1}{\sqrt{x^3+5}}dx < \int_b^\infty \frac{1}{\sqrt{x^3}}dx = \frac{2}{\sqrt{b}}.$$

So it suffices to choose b such that $\frac{2}{\sqrt{b}} < 0.01$ or $b > 40,000$, say for example $b = 45000$. In this case,

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx \approx \int_1^{45,000} \frac{1}{\sqrt{x^3+5}}dx = 1.69824.\blacksquare$$

Example 47.3
Investigate the convergence of $\int_4^\infty \frac{dx}{\ln x - 1}$.

Solution.
For $x \geq 4$ we know that $\ln x - 1 < \ln x < x$. Thus, $\frac{1}{\ln x - 1} > \frac{1}{x}$. Let $g(x) = \frac{1}{x}$ and $f(x) = \frac{1}{\ln x - 1}$. Thus, $0 < g(x) \leq f(x)$. Since $\int_4^\infty \frac{1}{x}dx = \int_1^\infty \frac{1}{x}dx - \int_1^4 \frac{1}{x}dx$ which is divergent since $\int_1^\infty \frac{1}{x}dx$ is divergent being a p-integral with $p = 1$. By the comparison test $\int_4^\infty \frac{dx}{\ln x - 1}$ is divergent.$\blacksquare$

Example 47.4
Investigate the convergence of the improper integral $\int_1^\infty \frac{\sin x + 3}{\sqrt{x}}dx$.

Solution.
We know that $-1 \leq \sin x \leq 1$. Thus $2 \leq \sin x + 3 \leq 4$. Since $x \geq 1$, we have $\frac{2}{\sqrt{x}} \leq \frac{\sin x + 3}{\sqrt{x}} \leq \frac{4}{\sqrt{x}}$. Note that the two integrals $\int_1^\infty \frac{2}{\sqrt{x}}dx$ and $\int_1^\infty \frac{4}{\sqrt{x}}dx$

are both divergent being a multiple of a p-integral with $p = \frac{1}{2} < 1$. If we let $g(x) = \frac{\sin x + 3}{\sqrt{x}}$ and $f(x) = \frac{4}{\sqrt{x}}$ then we have no conclusion since $\int_1^\infty g(x)dx$ may or may not converge and still $\int_1^\infty g(x)dx \le \int_1^\infty f(x)dx$. Now if we let $g(x) = \frac{2}{\sqrt{x}}$ and $f(x) = \frac{\sin x+3}{\sqrt{x}}$ then by the comparison test $\int_1^\infty \frac{\sin x+3}{\sqrt{x}}$ is divergent since $\int_1^\infty f(x)dx \ge \int_1^\infty g(x)dx$ and $\int_1^\infty g(x)dx$ is divergent.■

Example 47.5
Investigate the convergence of $\int_1^\infty e^{-\frac{1}{2}x^2}dx$.

Solution.
If $x \ge 2$ then $\frac{x}{2} \ge 1$. Multiply both sides of this inequality by $x \ge 2$ to obtain $\frac{1}{2}x^2 \ge x$. Now, multiply both sides of this last inequality by -1 to obtain $-\frac{1}{2}x^2 \le -x$ and therefore $e^{-\frac{1}{2}x^2} \le e^{-x}$ since the function e^x is an increasing function. Thus,

$$\int_1^\infty e^{-\frac{1}{2}x^2}dx = \int_1^2 e^{-\frac{1}{2}x^2}dx + \int_2^\infty e^{-\frac{1}{2}x^2}dx.$$

But

$$\int_1^2 e^{-\frac{1}{2}x^2}dx \approx 0.34$$

and

$$\int_2^\infty e^{-\frac{1}{2}x^2}dx \le \int_2^\infty e^{-x}dx \le \int_0^\infty e^{-x}dx$$

so since $\int_0^\infty e^{-x}dx$ is convergent, $\int_2^\infty e^{-\frac{1}{2}x^2}dx$ is convergent. In conclusion, $\int_1^\infty e^{-\frac{1}{2}x^2}dx$ is convergent.■

Sometimes it is laborious to find a convenient function $f(x)$ with $g(x) \le f(x)$, but we may know that $g(x)$ is no larger than a constant multiple of $f(x)$ for large enough x, and this is good enough. The most powerful test of this form in the course is this version of the **limit comparison test:**

Theorem 47.2
Let $f(x)$ and $g(x)$ be two positive and continuous functions on $[a, \infty)$.

(a)$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, or
(b)$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where L is a finite positive number, or
(c)$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then

(a) If $\int_a^\infty g(x)dx$ converges, then so does $\int_a^\infty f(x)dx$.
(b) $\int_a^\infty g(x)dx$ converges if and only if $\int_a^\infty f(x)dx$ does.
(c) If $\int_a^\infty g(x)dx$ diverges, then so does $\int_a^\infty f(x)dx$.

Proof.
(a) Suppose that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. Let $\epsilon > 0$ be given. Then there is a $b > a$ such that $\frac{f(x)}{g(x)} < \epsilon$ for all $x \geq b$. Thus, $f(x) < \epsilon g(x)$ for all $x \geq b$. By the comparison test, if $\int_a^\infty g(x)dx$ is convergent so does $\int_a^\infty f(x)dx$.
(b) Now, suppose that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where L is a finite positive constant. Let $\epsilon < L$. Then there is a constant $b > a$ such that for all $x \geq b$ we have

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon.$$

That is,

$$L - \epsilon < \frac{f(x)}{g(x)} < L + \epsilon.$$

Thus, for $x \geq b$ we have $(L - \epsilon)g(x) < f(x) < (L + \epsilon)g(x)$. Now the result follows from the comparison test.
(c) Finally, suppose that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. Then there is $b > a$ such that $\frac{f(x)}{g(x)} \geq 1$ for all $x \geq b$. That is, $g(x) \leq f(x)$ for all $x \geq b$. Therefore, if $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.■

Remark 47.1
The Comparison Test and Limit Comparison Test also apply, modified as appropriate, to other types of improper integrals.

Example 47.6
Show that the improper integral $\int_1^\infty \frac{1}{1+x^2}dx$ is convergent.

Solution.
Since the integral $\int_1^\infty \frac{dx}{x^2}$ is convergent (p-integral with $p = 2 > 1$) and since $\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x^2+1} = 1$, by the limit comparison test (Theorem 47.2 (b)) we have $\int_1^\infty \frac{dx}{x^2+1}$ is also convergent.■