EXERCISE SET III : RECCURENCE

Problem 1. Find the general solution for each of the following relations.
  a) \(4a_n - 5a_{n-1} = 0, \ n \geq 1\);
  b) \(3a_{n+1} - 4a_n = 0, \ n \geq 0, \ a_1 = 5\);
  c) \(2a_n - 3a_{n-1} = 0, \ n \geq 1, \ a_4 = 81\);
  d) \(a_n = 5a_{n-1} + 6a_{n-2}, \ n \geq 2, \ a_0 = 1, \ a_1 = 3\);
  e) \(3a_{n+1} = 2a_n + a_{n-1}, \ n \geq 1, \ a_0 = 7, \ a_1 = 3\);
  f) \(a_n - 6a_{n-1} + 9a_{n-2} = 0, \ n \geq 2, \ a_0 = 5, \ a_1 = 12\);
  g) \(a_{n+2} + a_n = 0, \ n \geq 0, \ a_0 = 0, \ a_1 = 3\);
  h) \(a_n + 2a_{n-1} + 2a_{n-2} = 0, \ n \geq 2, \ a_0 = 1, \ a_1 = 3\).
(The final answer should not involve complex numbers.)

Problem 2. The number of bacteria in a culture is 1000 (approximately) and this number increases 250% every two hours. Use a recurrence relation to determine the number of bacteria after one day.

Problem 3. Laura invested $100 at 6% interest compounded quarterly. How many months must she wait for her money to double?

Problem 4. Motorcycles and compact cars can be parked in a row of \(n\) spaces, so that each motorcycle requires one space and each compact car needs two. Suppose that all cycles are identical in appearance, as are the cars. Find and solve a recurrence relation for the number of ways to park so that all the \(n\) spaces are filled up.

Problem 5. For \(n \geq 0\), let \(a_n\) count the number of ways a sequence of 1’s and 2’s will sum up to \(n\). (For example, \(a_3 = 3\), because 3 can be presented in three ways: \(1 + 1 + 1\), \(1 + 2\), and \(2 + 1\).) Find and solve a recurrence relation for \(a_n\).

Problem 6. For \(n \geq 1\), let \(a_n\) be the number of ways to present \(n\) as an ordered sum of positive integers, so that each summand is at least two. (For example, \(a_5 = 3\), because 5 can be presented in three ways: \(5\), \(3 + 2\), and \(2 + 3\).) Find and solve a recurrence relation for \(a_n\).

Problem 7. Start with one pair of rabbits and suppose that each pair produces one new pair in each of the next two generations and then dies. Find the number \(f_n\) of pairs belonging to the \(n\)-th generation.

Problem 8. The Lucas numbers \(L_n\) are defined by \(L_1 = 1, \ L_2 = 3, \ L_n = L_{n-1} + L_{n-2}\). Obtain a formula for \(L_n\).

Problem 9. Solve the recurrence relation \(a_{n+2} = a_n a_{n+1}\).
Problem 10. Solve the recurrence relation \( a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0 \), where \( n \geq 0 \), \( a_0 = 4 \), \( a_1 = 13 \).

Problem 11. Determine the constants \( b \) and \( c \) if \( a_n = c_1 + c_2(7^n) \), \( n \geq 0 \), is the general solution of the relation \( a_{n+2} + ba_{n+1} + ca_n = 0 \).

Problem 12. Solve each of the following recurrence relations.

a) \( a_{n+1} - 2a_n = 5 \), \( a_0 = 1 \).

b) \( a_{n+1} - 2a_n = 2^n \), \( a_0 = 1 \).

c) \( a_{n+1} - a_n = 2n + 3 \), \( a_0 = 1 \).

d) \( a_{n+1} - a_n = 3n^2 - n \), \( a_0 = 3 \).

e) \( a_n = 4a_{n-1} - 3a_{n-2} + 2^n \), \( a_1 = 1 \), \( a_2 = 11 \).

f) \( a_{n+2} + 3a_{n+1} + 2a_n = 3^n \), \( a_0 = 0 \), \( a_1 = 1 \).

g) \( a_{n+2} + 4a_{n+1} + 4a_n = 7 \), \( a_0 = 1 \), \( a_1 = 2 \).

h) \( a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n) \), \( a_0 = 1 \), \( a_1 = 4 \).

i) \( a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 3 + 5n \), \( a_0 = 1 \), \( a_1 = 4 \).

j) \( a_{n+2} - 5a_{n+1} + 6a_n = 7n \), \( a_0 = a_1 = 1 \).