1. Prove that in a topological space the union of a finite family of compact sets is compact.

2. Consider a set $X$ and $p \in X$. Let $A \subseteq X$ be declared open if either $p \notin A$ or $p \in A$ and $X - A$ is finite.
   (A) Prove that these open sets constitute a topology on $X$.
   (B) Prove that $X$ is compact with respect to this topology.

3. (A) Prove that in a regular space the closure of a compact set is compact.
   (B) Give an example of a topological space which has a proper compact subset whereof
   the closure is not compact.

4. (A) Let $X, Y$ be topological spaces. Given compact $K \subseteq X$, $y \in Y$ and an open
   $W \subseteq X \times Y$ such that $K \times \{y\} \subseteq W$, prove that $y$ has a
   neighbourhood $V \subseteq Y$ such that $K \times V \subseteq A$.
   (B) If $X$ is compact, prove that the projection $p : X \times Y \longrightarrow Y$ onto
   the second component is a closed map.

5. Let $X$ be a Hausdorff topological space.
   (A) Prove that an infinite compact $A \subseteq X$ contains at least one accumulation point
   of itself.
   (B) Suppose that there exists a (not necessarily continuous!) map
   $f : X \longrightarrow \mathbb{R} - \{0\}$
   such that
   $$\lim_{{x \to p}} \frac{1}{f(x)} = 0$$
   for each accumulation point $p$ of $X$. Prove that $K \cap f^{-1}([a, b])$ is a finite set
   for every compact $K \subseteq X$ and $a, b \in \mathbb{R}$. 
(C) Prove that every infinite compact subset of $X$ is countable.

6. Prove that the product of a Lindelöf space with a compact space is Lindelöf.

7. (A) Let $\mathcal{T}$, $\mathcal{T}'$ be topologies on $X$ and $\mathcal{T} \subseteq \mathcal{T}'$. (In other words, $\mathcal{T}'$ is stronger, or finer than $\mathcal{T}$.) If $A \subseteq X$ is compact with respect to $\mathcal{T}'$, prove that $A \subseteq X$ is also compact with respect to $\mathcal{T}$.

   (B) If $A \subseteq \mathbb{R}$ is compact in $\mathbb{R}_{sor}$, prove that $A \subseteq \mathbb{R}$ is compact in $\mathbb{R}$, as well.

   (C) For any $a, b \in \mathbb{R}$ with $a < b$, prove that $(a, b)$, $[a, b)$, $[a, b]$, $]a, b]$ are not compact in $\mathbb{R}_{sor}$.

   (D) If $A \subseteq \mathbb{R}$ is compact in $\mathbb{R}_{sor}$, prove that $A \subseteq \mathbb{R}$ is bounded, $A$ is closed in $\mathbb{R}_{sor}$ and contains no strictly increasing sequence.

   (E) If $A \subseteq \mathbb{R}$ is not compact in $\mathbb{R}_{sor}$, prove that $A$ contains a strictly increasing sequence. (Hint: There exists a countable (why?) open cover $\mathcal{A} = \{V_i\}_{i=1}^{\infty}$ of $A$ with no finite subcover. Choose $x_n = \inf (A - \bigcup_{i=1}^{n} V_i)$.) Conclude that $A$ is compact in $\mathbb{R}_{sor}$ if it is closed in $\mathbb{R}_{sor}$, bounded and contains no strictly increasing sequence. 1

   (F) If $A \subseteq \mathbb{R}$ is compact in $\mathbb{R}_{sor}$, prove that for any $a, b \in A$ with $a < b$, there exists $m \in \mathbb{R}$ with $a < m < b$ such that $A \cap (m, b) = \emptyset$.

   (G) Prove that $A \subseteq \mathbb{R}$ is compact in $\mathbb{R}_{sor}$ if $A$ is bounded, $A$ is closed in $\mathbb{R}_{sor}$ and $-A = \{x \in \mathbb{R} \mid -x \in A\}$ is discrete.

   (H) Prove that a compact subset of $\mathbb{R}_{sor}$ is countable.

8. Let $X = \mathbb{Z} \times \mathbb{Z}$ and let $\tau$ consist of sets $A \subseteq X$ with the property that either $(0, 0) \notin A$ or there exists $N \subseteq A$ with $(0, 0) \in N$ such that $(\{m\} \times \mathbb{Z}) \cap (X - N)$ is finite for all except finitely many $m \in \mathbb{Z}$.

   (A) Prove that $\tau$ constitutes a topology on $X$. 2

   (B) Prove that $X$ 3 is Hausdorff and regular. Conclude that $X$ is in fact $T_4$. 4

   (C) Prove that $(0, 0) \in X$ has no compact neighbourhood. Conclude that $X - \{(0, 0)\}$ is dense.

   (D) Prove that every neighbourhood of $(0, 0) \in X$ is closed.

   (E) Prove that no sequence in $X - \{(0, 0)\}$ converges to $(0, 0)$.

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1M. S. Espelie, J. E. Joseph – Compact subsets of the Sorgenfrey line, Mathematics Magazine 49 (1976) 250-251

2The system $(X, \tau)$ is the so called Arens-Fort Space. See: Counterexamples in Topology by L. A. Steen, J. A. Seebach, jr.

3$X$ henceforth designates the topological space $(X, \tau)$.

4Being countable, $X$ is trivially Lindelöf.
(F) Let the sequence $(y_n)_{n \in \mathbb{N}}$ be an enumeration of $X - \{(0,0)\}$. In other words, let
\[ \{y_n \mid n \in \mathbb{N}\} = X - \{(0,0)\}. \]
Prove that $y_n$ has a subnet that converging to $(0,0)$ which is not a subsequence.

9. A subset of a topological space is referred to as a $G_\delta$-set if it is the intersection of a countable family of open sets. A topological space is said to be perfectly normal if it is normal and every closed subset thereof is a $G_\delta$-set.

(A) Given a topological space $X$ and a continuous map $f : X \rightarrow \mathbb{R}$, prove that $f^{-1}(t)$ is a $G_\delta$-set for every $t \in \mathbb{R}$.

(B) If $Y$ is a normal space, prove that for every closed $G_\delta$-set $B \subseteq Y$, there exists a continuous function $g : Y \rightarrow \mathbb{R}$ such that $B = g^{-1}(0)$.

(C) Prove that a pseudometrisable space is perfectly normal.

(D) Is $[0,1]^{\mathbb{R}}$ perfectly normal?

\footnote{Consider singletons!}