1. Given $i = 1, 2, 3$, prove that the product of $T_i$ spaces is a $T_i$ space.

2. Prove that a countable product of $C_{II}$ spaces is $C_{II}$. Exhibit a family of $C_{II}$ spaces whereof the product is not $C_{II}$.

3. Prove that a countable product of separable spaces is separable. Exhibit a family of separable spaces whereof the product is not separable.

4. Given topological spaces $X_\alpha, \alpha \in \Lambda$, and subsets $A_\alpha \subseteq X_\alpha$, prove that

$$\text{Cl} \left( \prod_{\alpha \in \Lambda} A_\alpha \right) = \prod_{\alpha \in \Lambda} \text{Cl}_{X_\alpha}(A_\alpha).$$

5. Given topological spaces $X, Y$ and subsets $A \subseteq X, B \subseteq Y$ prove that

$$\text{In}_{X \times Y}(A \times B) = \text{In}_X(A) \times \text{In}_Y(B).$$

What can you say about the boundary of $A \times B$ in $X \times Y$?

6. Given a set $X$, the set of subsets of $X$ (denoted by $\mathcal{P}(X)$ or $2^X$) can as usual be identified with the product space $\{0, 1\}^X$ (which can be regarded, as usual, as the set of functions $f : X \rightarrow \{0, 1\}$). We consider $\{0, 1\}^X$ to have the product topology by assuming that $\{0, 1\}$ has the discrete topology. Let the map $\iota : X \rightarrow \mathcal{P}(X)$ be defined by $\iota(a) = \{a\}$ for any $a \in X$.

(A) Prove that $\emptyset \in \mathcal{P}(X)$ is the unique accumulation point of $\mathfrak{A} = \iota(X) \subseteq \mathcal{P}(X)$.

(B) Is $\mathfrak{A}$ a closed subset of $\mathcal{P}(X)$?
7. Let $X,Y$ be topological spaces, and $q : X \longrightarrow Y$ a quotient map.
   
   (A) Prove that $L \subseteq Y$ is closed iff $q^{-1}(L) \subseteq X$ is closed.
   
   (B) Prove that $Y$ is a $T_1$ space iff $q^{-1}\{y\}$ is closed, for every $y \in Y$.
   
   (C) Consider the line with the double origin $Y = (\mathbb{R} - \{0\}) \cup \{0_1, 0_2\}$ with its usual pseudometric and the usual quotient map $q : X = \mathbb{R} \times \{1, 2\} \longrightarrow Y$. Find a point $b \in Y$ such that $q^{-1}\{b\} \subseteq X$ is not closed.

8. Let $X,Y$ be topological spaces, and $q : X \longrightarrow Y$ a quotient map. Let
   
   $$\mathcal{R} = \{(x,x') \in X \times X \mid q(x) = q(x')\} \subseteq X \times X$$
   
   (A) If $Y$ is Hausdorff, prove that $\mathcal{R}$ is a closed subset of $X \times X$.
   
   (B) If $\mathcal{R}$ is a closed subset of $X \times X$ and $q : X \longrightarrow Y$ is open, prove that $Y$ is Hausdorff.

9. Let $X,Y$ be topological spaces, and $q : X \longrightarrow Y$ a quotient map. For any $A \subseteq X$, let
   
   $$\mathcal{G}[A] = \{x \in X \mid \exists a \in A, q(x) = q(a)\} \subseteq X \times X$$
   
   (A) Prove that $q$ is a closed map iff $\overline{\mathcal{G}[A]} \subseteq \mathcal{G}[\overline{A}]$ for any $A \subseteq X$.
   
   (B) Prove that $q$ is an open map iff $\mathcal{G}[A^o] \subseteq \mathcal{G}[A]^o$ for any $A \subseteq X$.