# MATH 373 - GEOMETRY I

## FINAL EXAMINATION

<table>
<thead>
<tr>
<th>FAMILY NAME</th>
<th>OTHER NAMES</th>
<th>GRADE</th>
</tr>
</thead>
<tbody>
<tr>
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11th January 2011. Duration: 2.5 hours.

Six questions: $8 + 8, 4 + 8 + 4 + 4, 8 + 4 + 8, 8, 4 + 4 + 4 + 4 + 8$

### Solutions
1. Given a triangle $ABC$, consider $P \in BC - \{B, C\}$, $Q \in CA - \{C, A\}$, $R \in AB - \{A, B\}$ such that $AP$, $BQ$, $CR$ are concurrent. Let $QR$, $RP$, $PQ$ meet $BC$, $CA$, $AB$ in $X$, $Y$, $Z$ respectively. Prove that
(A) $X$, $Y$, $Z$ are collinear.
(B) $AP$, $BY$, $CZ$ are concurrent or parallel.

\[
\text{(A) } \text{"Ceva" in } ABC \\
\frac{PB}{PC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB} = -1 \quad \text{since } AP, BQ, CR \text{ are concurrent.}
\]

By Menelaus in $ABC$ with $QR$:

\[
\frac{Xb}{XC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB} = +1
\]

hence \[
\frac{Xb}{XC} = \left(\frac{QC}{QA} \cdot \frac{RA}{RB}\right)^{-1}
\]
We can similarly compute \[
\frac{Yc}{YA} \text{ and } \frac{Za}{ZB}
\]
and

\[
\frac{Xb}{XC} \cdot \frac{Yc}{YA} \cdot \frac{Za}{ZB} = \frac{PB}{PC} \left(\frac{QC}{QA} \cdot \frac{RA}{RB}\right)^{-2} = 1
\]
Consequently by "Menelaus", $X$, $Y$, $Z$ are collinear.

\[
\text{(B) As } \frac{PB}{PC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \frac{PB}{PC} \left(\frac{RA}{RB} \cdot \frac{PB}{PC}\right)^{-1} \left(\frac{PB}{PC} \cdot \frac{QC}{QA}\right)^{-1}
\]

\[
= \left(\frac{PB}{PC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB}\right)^{-1} = -1
\]
we conclude by "Ceva", that $AP$, $BY$, $CZ$ are concurrent or parallel.
2. Let $ABC$ be a triangle with orthocenter $H$, incenter $I$ and centroid $G$. Given a point $X$, let $G_a$, $G_b$, $G_c$ be the centroids of $XBC$, $XCA$, $XAB$ respectively.

(A) Prove that $G_bG_c$ is parallel to $BC$ and $GG_a$ is parallel to $AX$.

(B) How can we choose $\alpha, \beta \in \mathbb{R}$ so that

$$\text{Hom}(X, \beta) \circ \text{Hom}(G, \alpha)(ABC) = G_aG_bG_c?$$

(C) Show that the centroid of $G_aG_bG_c$ is on the line $XG$. Where is it exactly?

(D) If $X = H$ prove that $G$ is the orthocenter of the triangle $G_aG_bG_c$.

(E) If $X = I$ which remarkable point of the triangle $G_aG_bG_c$ is the point $G$?

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**Diagram:**

Let $A'$, $b'$, $c'$ be the respective midpoints of $[B, C]$, $[C, A]$, $[A, B]$. Since

$$\frac{G_bX}{G_b'b'} = \frac{G_cX}{G_c'c'} = -\frac{1}{2},$$

we find that

$$G_bG_c \parallel b'c' \parallel BC.$$  

Similarly

$$\frac{G_A'}{G_A} = -\frac{1}{2}.$$

Hence $G_Ga \parallel AX$.

(B) Since $\text{Hom}(X, \alpha) \circ \text{Hom}(G, \beta) : [A' \rightarrow G_a]$ 

$$B' \rightarrow G_b,$$

$$C' \rightarrow G_c.$$  

Clearly $\alpha = -\frac{1}{2}$, $\beta = \frac{2}{3}$.

(C) The centroid of $G_aG_bG_c$ is $Z = \text{Hom}(X, \frac{2}{3}) \circ \text{Hom}(G, -\frac{1}{2})(G)$

$$= \text{Hom}(X, \frac{2}{3})(G). \quad \therefore G \text{ is on } XG$.

(and indeed $\frac{ZX}{GX} = \frac{2}{3}$)

(D) If $X = H$, then $G_bG_c \parallel BC \perp AH \parallel GG_a$.

Similarly $G_Gb \perp G_cG_a$. \therefore $G$ is the orthocenter of $G_aG_bG_c$.

(E) $AI$ bisects the angle $X$ $BAC$. As $GA \parallel G_cG_a$, $BA \parallel G_bG_a$ and $IA \parallel G_Ga$

we conclude that $G_Ga$ bisects the angle $X$ $G_bG_aG_c$ internally. Similarly...

Therefore $G$ is the incenter of $G_aG_bG_c$. 

3. Consider a positively oriented triangle $ABC$ and positively oriented equilateral triangles $BSC$, $CTA$, $AUB$.

(A) Let $AS$ intersect $BC$ in $X$. Comparing the (oriented) areas of the triangles $A_sPB$ and $A_sPC$ compute $XB/XC$ in terms of the angles $B$ and $C$.

(B) Prove that $AS$, $BT$, $CU$ are concurrent.

(C) Let $M_a$, $M_b$, $M_c$ be the centers of the equilateral triangles $BSC$, $CTA$, $AUB$. First proving

$$\text{Rot}(M_b, 2\pi/3) \circ \text{Rot}(M_c, 2\pi/3) = \text{Rot}(M_a, -2\pi/3)$$

or otherwise, show that $M_aM_bM_c$ is an equilateral triangle.

(c) Note that

$$\text{Rot}(M_a, 2\pi/3) \circ \text{Rot}(M_b, 2\pi/3) \circ \text{Rot}(M_c, 2\pi/3)$$

leaves $B$ invariant. It is also a translation since

$$2\pi/3 + 2\pi/3 + 2\pi/3 = 0 \pmod{2\pi}$$

It follows that the line $u$ through $M_c$ and $v$ through $M_b$ with

$$\mathcal{X}(u, M_b, M_c) = \mathcal{X}(M_b, M_c, v) = \frac{\pi}{3}$$

intersect in $M_a$.
4. Given triangle $XYZ$, let $T$ be a point on $YZ$ and let $F, G$ be points on $XY$. Let $FT$ meet $XZ$ in $H$, $GT$ meet $XZ$ in $K$, $ZF$ meet $XT$ in $M$ and $ZG$ meet $XT$ in $N$. Prove that $KM$ and $HN$ intersect on $XY$ or $KM$, $HN$, $XY$ are parallel.

Note: $1'2' = N \quad 1'2 = H$$2'3' = F \quad 2'3 = G$$3'4' = K \quad 3'1 = M$.

By the dual of the Pappus theorem, $HN \land KM \land FG = XY$ are either concurrent or parallel.
5. Let \( \varphi \) be an ellipse. Prove that the set of points from which the tangents to \( \varphi \) are perpendicular to one another, is a circle.

\[
|OX|^2 = |ZF|^2 = |FM|^2 + |FN|^2 - |FF'|^2
\]

\[= 8a^2 - 4c^2 = \text{constant!}\]
6. (A) What is the image of the circles \((x - 1)^2 + y^2 = 1\) and \((x - 2)^2 + y^2 = 4\) under the inversion \(\text{Inv}(0, 0, 1)\)?

(B) What is the image of the lines \(x = 3\) and \(x = 5\) under the inversion \(\text{Inv}(0, 0, 30)\)?

(C) What is the image of the circles \((x - 1)^2 + y^2 = 1\) and \(x^2 + (y - 1)^2 = 1\) under the inversion \(\text{Inv}(0, 0, 6)\)?

(D) What is the image of the circle \((x - 1)^2 + y^2 = 1\) under the inversion \(\text{Inv}(-1, 0, 6)\)?

(E) Find the homothety centers of the circles in (D).

\[\begin{align*}
(A) & \quad \text{Graph showing the circle and its image under inversion.} \\
(B) & \quad \text{Graph showing the line and its image under inversion.} \\
(C) & \quad \text{Graph showing the circle and its image under inversion.} \\
(D) & \quad \text{Graph showing the circle and its image under inversion.} \\
(E) & \quad \text{One of the homothety centers (the external one) is } (-1, 0) \text{ since it is the inversion center. The other is } (x, 0) \text{ with } \\
& \quad \frac{x - 1}{x - 3} = -\frac{1}{2} \quad \rightarrow \quad 2x - 2 = -(x - 3) \\
& \quad x = \frac{5}{3} \\
& \quad \therefore P_{\text{int}} = \left(\frac{5}{3}, 0\right) \\
\end{align*}\]