Geometry I

MATH 373

SECOND MIDTERM

(Duration: 100 mins.)

24th December 1999

\[ 7 + 8 + 10 \], \[ 15 + 10 \], \[ 15 + 15 + (10 + 10) \]

1.

(a) What is the image of the line \(3x - y - 4\) under \(T_{[1,-3]}\)?

(b) What is the image of the circle \(x^2 + y^2 - 5y + 1 = 0\) under \(R_{\frac{x}{k}}\), where \(k\) is the line \(y = 2x\)?

(c) What is the image of the line \(x = 1\) under \(Rot((-1,0), \pi/3)\)?

(a) The line in question contains the point \((0, -4)\) which moves into \((1, -7)\) under \(T_{[1,-3]}\). The line required is

\[3x - y = 3 \cdot 1 - (-7) = 10\]

\[3x - y - 10 = 0\]

(b) The circle in question is the circle of center \((0, \frac{5}{2})\), and radius \(\frac{\sqrt{21}}{2}\).
By a simple inspection it is found that
\[ \text{Ref } (0, \frac{5}{2}) = (2, \frac{3}{2}) \]
\[ \therefore \text{ The image of the circle is} \]
\[ (x - 2)^2 + (y - \frac{3}{2})^2 = \frac{21}{4} \]
or:
\[ x^2 + y^2 - 4x - 3y + 1 = 0 \]

The point \((1,0)\) moves into \((0, \sqrt{3})\) under
\[ \text{Rot } (-1,0, \frac{\pi}{3}) \]
Under the same rotation \(x = 1\) is turned into a line of the form
\[ x + \sqrt{3} y = m \]
\[ \therefore \text{ Therefore, the required line is} \]
\[ x + \sqrt{3} y = 0 + \sqrt{3} \cdot \sqrt{3} = 3 \]
2.

Let $E, (E), R$ and $\Delta$ denote the center of the nine point circle, the nine point circle, the circumradius and the area of the triangle $ABC$ respectively.

(a) Compute the power of $A$ with respect to $(E)$.

(b) Show that

$$|AE| = \sqrt{\Delta \cot A + \frac{R^2}{4}}$$

\[\begin{align*}
(a) & \quad (E) \text{ goes through the midpoint } C' \text{ of } [A, B] \text{ and the foot } C \text{ of the altitude through } C. \\
& \quad \text{Consequently the required power is} \\
& \quad AC'. AC = \frac{b}{2} \cdot c \cdot \cos A \\
& \quad = \frac{\Delta}{\sin A} \cos A \\
& \quad = \Delta \cot A
\end{align*}\]

(b) Obvious from

$$|AE|^2 - \frac{R^2}{4} = \text{the power of } A \quad \text{with respect to } (E).$$
In solving the following problems the student is not allowed to use the word "infinity" or the sign $\infty$.

(a) State and prove the first theorem of Poncelet for a parabola.

(b) State and prove the second theorem of Poncelet for a parabola.

(c) On a parabola of focus $F$, consider points $X, Y$ such that $F \in XY$. Let the tangents to the parabola at $X, Y$ intersect in $P$. Prove that $XP \perp YP$ and $FP \perp XY$.

\[
(a) \text{ If } \text{tangent } t, t' \text{ to a parabola of focus } F, \text{ directrix } d \text{ intersect in } P \text{ then:}
\]
\[
\xi(t', k) = \xi(PF, t)
\]
where $k$ is any line with $k \perp d$.

\[
\text{Proof:}
\]
\[
\xi(t', k) = 4 = 3 (P, W, Q, N \text{ concyclic})
\]
\[
= 2 = 1 = \xi(PF, t)
\]
\[
(P, Q, R, F \text{ concyclic})
\]
(b) If the tangents to a parabola of focus \( F \) at \( X \in \gamma, Y \in \gamma \) intersect in \( P \), then
\[
\angle (FX,FP) = \angle (FP, FY)
\]

**Proof:**
\[
\angle (FX,FP) = \gamma = \pi - 5 = \pi - 6
\]
\[
= 8 = \angle (FP, FY).
\]

(c)

By Poncelet 2,
\[ w = \beta, \text{ as } \alpha + \beta = \pi \]
we conclude \[ w = \beta = \frac{\pi}{2} \]
Hence \( PF \perp XY \)

On the other hand
\[ \gamma = 5 = 2 = 1 \]

Poncelet 1
\[ = \frac{\pi}{2} - 4 \]

which shows that \( PXY \) is a right triangle...