1. Write down the equations of the osculating, normal and rectifying planes and evaluate the curvature and the torsion of the following curves:
   (A) $\mathbf{r}(t) = (t - \sin t, 1 - \cos t, t)$
   (B) $\mathbf{r}(t) = (\cos t + \sin t, \sin t + \cos t, \sin 2t)$
   (B) $\mathbf{r}(t) = (3t - t^3, 3t^2, 3t + t^3)$

2. Prove that at each point of the twisted cubic the unique plane that contains the tangent line and contains exactly one point of the curve is the osculating plane.

3. Prove that the curve $\mathbf{r} : (0, \infty) \rightarrow \mathbb{R}^3$ defined by
   $$\mathbf{r}(t) = \left(t, 1 + \frac{1}{t}, \frac{1}{t} - t\right)$$
   is coplanar.

4. Prove that each involute of the circular helix is coplanar.

5. Prove that in order for the principal normal of $\mathbf{r}(t) = (\varphi(t), \psi(t), t)$ at each point of this curve to be parallel to the $xy$-plane, it is necessary and sufficient that the quantity $(\dot{\varphi}(t))^2 + (\dot{\psi}(t))^2$ is a constant.

6. Suppose that $\theta$ and $\varphi$ are the respective angles which the tangent and the binormal of a curve with curvature $\kappa$ and torsion $\tau$ make with a fixed direction. Prove that
   $$\dot{\theta} \sin \theta : \dot{\varphi} \sin \varphi = -\kappa : \tau.$$

7. Consider a curve $\mathbf{r} : J \rightarrow \mathbb{R}^n$, where $n = 2$ or $3$ with nowhere vanishing curvature $\kappa$. Let $\mathbf{m} : J \rightarrow \mathbb{R}^n$ be defined by
   $$\mathbf{m}(t) = \mathbf{r}(t) + \frac{1}{\kappa(t)} \mathbf{N}(t).$$
   for all $t \in J$ where $\mathbf{N}$ is the normal (or the principal normal) field along $\mathbf{m}$.
   (A) Let $n = 2$. Prove that $\mathbf{m}$ is regular iff $\kappa$ vanishes nowhere.\(^1\)

\(^1\)For each $t \in J$, the point $\mathbf{m}(t)$ is called the **centre of curvature** of $\mathbf{r}$ at $\mathbf{r}(t)$. The curve $\mathbf{m}$, that is, the curve traced out by the center of curvature of $\mathbf{r}$ is called the **evolute** of $\mathbf{r}$.
(B) Prove that a regular curve in $\mathbb{R}^2$ with nowhere vanishing non-constant curvature is the evolute of any of its involutes.

8. Given a curve $\mathbf{r} : J \rightarrow \mathbb{R}^3$ with Frenet frame $(T, N, B)$, prove that there exists a vector field $D : J \rightarrow \mathbb{R}^3$ along $\mathbf{r}$ such that

$$
\begin{align*}
T' &= D \times T \\
N' &= D \times N \\
B' &= D \times B
\end{align*}
$$

\[\text{If the triple } (T, N, B) \text{ is regarded as a rigid body moving along the curve with unit speed, its instantaneous motion has linear velocity } T \text{ and angular velocity } D \text{ which is called the } \textbf{Darboux Vector} \text{ associated with the curve in question.}\]