1. Sketch roughly the following curves:

(A) \( \mathbf{r}(t) = (t, 5 + \sin t) \)

(B) \( \mathbf{r}(t) = (t, \cosh t) \) \diamond \text{Catenary} \diamond

(C) \( \mathbf{r}(t) = (\sin (3t) \cos t, \sin (3t) \sin t) \)

(D) \( \mathbf{r}(t) = (\text{sgn}(\cos t) \cos^2 t, \text{sgn}(\sin t) \sin^2 t) \)

(E) \( \mathbf{r}(t) = (\text{sgn}(\cos t) \cos^2 t, \text{sgn}(\sin t) \sin^2 t, t) \)

(F) \( \mathbf{r}(t) = \left( \cos \left( \frac{1}{t} \right), \sin \left( \frac{1}{t} \right), t \right) \) where \( t > 0 \).

2. Prove that all normal planes of the curve

\[ \mathbf{r}(t) = (\sin^2 t, \sin t \cos t, \cos t) \]

go through the point \((0, 0, 0)\).

3. Consider the curve

\[ \mathbf{r}(t) = \left( e^t \cos t, e^t \sin t \right) . \]

(A) Sketch this curve roughly.

(B) Let \( O = (0, 0) \). For any point \( P \) on this curve, show that the angle between the line \( OP \) and the tangent line at \( P \) is a constant independent of the position of \( P \) on the curve.

(C) For any points \( P, Q \) on the curve, if the line \( PQ \) goes through \( O \), prove that the tangent lines at \( P, Q \) are parallel. \diamond \text{Equiangular spiral} \diamond

4. Prove that through each tangent line of the “twisted cubic”, there passes exactly one plane that intersects the curve in question in exactly one point.

5. (A) Prove that a curve \( \mathbf{r} \) in \( \mathbb{R}^3 \) lies on a sphere of center \( M \) iff

\[ \mathbf{\dot{r}} \cdot (\mathbf{r} - M) = 0 . \]

(B) Let \( \mathbf{r} \) be a curve in \( \mathbb{R}^3 \). If every normal plane of \( \mathbf{r} \) contains the point \( P \in \mathbb{R}^3 \), prove that \( \mathbf{r} \) lies on a sphere of center \( P \).
6. Remember that the “curve $r = r(\theta)$” of elementary calculus stands for the curve

$$t(t) = \left(r(t) \cos t, r(t) \sin t\right).$$

(A) Prove that $r = r(\theta)$ represents a regular curve iff

$$r^2 + \dot{r}^2 \neq 0.$$

(B) Prove that $r = r(\theta)$ is regular curve for the parameter value $\theta = \alpha$ unless $r(\theta) = 0$ in some neighbourhood of $\alpha$.

(C) Prove that the unit tangent field of $r = r(\theta)$ is of the form

$$T(t) = \frac{rE + r\dot{E}}{\sqrt{r^2 + \dot{r}^2}}$$

where

$$E(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

7. The circle $x^2 + y^2 - 2y = 0$ rolls without slipping on the $x$-axis in the positive direction, its center traversing unit distance in unit time. Prove that a point fixed on the moving circle and occupying the position $(0, 0)$ at $t = 0$ describes the curve

$$t(t) = \left(t - \sin t, 1 - \cos t\right).$$

For which values of $t$ does this curve fail to be regular? 📌 Cycloid 📌

8. A circle of radius $b$ rolls without slipping on circle of radius $a$. Show that that a point fixed on the moving circle describes - by a suitable choice of the configuration in $\mathbb{R}^2$ - the curve

$$t(t) = \left((a + b) \cos t - b \cos \left(\left(1 + \frac{a}{b}\right)t\right), (a + b) \sin t - b \sin \left(\left(1 + \frac{a}{b}\right)t\right)\right).$$

For which values of $t$ does this curve fail to be regular? 📌 Epicycloid 📌