Reparametrisation of Curves

From a formal point of view, two curves \( r_1, r_2 \) are to be understood as different objects unless they are identical as abstract maps. However, there are circumstances under which two curves defined by different formulae are clearly seen to have the same geometric portent.

**EXAMPLE 1:** Consider \( r_1 : (0, 1) \rightarrow \mathbb{R}^2 \) where
\[
r_1(t) = (t, t^2)
\]
and \( r_2 : (0, \pi/2) \rightarrow \mathbb{R}^2 \) where
\[
r_2(t) = \left( \sin \theta, (\sin \theta)^2 \right).
\]

By falling back on the intuitive content of the concept of a curve as a directed shape traced out by a point in motion, it is seen that although they are not identical, the maps \( r_1, r_2 \) describe very much the same geometry.

**DEFINITION 1:** Given open intervals \( J_1, J_2 \), a bijective map \( \rho : J_2 \rightarrow J_1 \) of class \( C^k, k \geq 1 \) with derivative vanishing nowhere on \( J_1 \) will be called a **reparametrisation** of class \( C^k \).

Since \( J_1 \) is an interval, hence connected, either \( \dot{\rho}(\theta) > 0 \) for all \( \theta \in J_2 \) or \( \dot{\rho}(\theta) < 0 \) for all \( \theta \in J_1 \). In the former case \( \rho \) will be said to be **orientation preserving**, in the latter **orientation reversing**.

Unless explicitly qualified otherwise, we shall make use of orientation preserving and smooth reparametrisations only.

**DEFINITION 2:** A curve \( r_2 : J_2 \rightarrow \mathbb{R}^n \) is said to be **obtained** from \( r_1 \) by **reparametrisation** \( \rho : J_2 \rightarrow J_1 \) if
\[
r_2 = r_1 \circ \rho,
\]
or equivalently \( r_2(\theta) = r_1(\rho(\theta)) \) for all \( \theta \in J_2 \).
In all simplicity, this means that in the formulae defining \( r_2 \) are obtained by substituting \( \rho(\theta) \) for \( t \) in the formulae defining \( r_1 \).

**REMARK 1:** It can be routinely checked that in Example 1 the curve \( r_2 \) is obtained from \( r_1 \) by the reparametrisation \( \rho : (0, \pi/2) \rightarrow (0, 1) \) where \( \rho(\theta) = \sin \theta \).

**EXAMPLE 2:** Consider \( r_1 : \mathbb{R} \rightarrow \mathbb{R}^2 \) where
\[
r_1(t) = \left( \frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2} \right)
\]
and \( r_2 : (-\pi, \pi) \rightarrow \mathbb{R}^2 \) where
\[
r_2(\theta) = (\sin \theta, \cos \theta).
\]
Again, both \( r_1 \) and \( r_2 \) will be seen to describe the same geometric object, namely the clockwise oriented circular segment \( S^1 - \{(0, -1)\} \). It can be checked that \( r_2 = r_1 \circ \rho \) where \( \rho : (-\pi, \pi) \rightarrow \mathbb{R} \) with \( \rho(\theta) = \tan (\theta/2) \).

**REMARK 2:** The function \( r_1 \) of the Example 2 which establishes a bijection between \( \mathbb{R} \) and \( S^1 - \{(0, -1)\} \) originates from a simple geometric construction: For each \( t \in \mathbb{R} \), let the straight line joining \((0, -1)\) and \((t, 0)\) intersect \( S^1 \) for a second time in \((x, y)\) : Clearly \( x : 1 + y = t : 1 \) hence
\[
\begin{align*}
  t &= \frac{x}{1 + y} &
  t^2 &= \frac{x^2}{(1 + y)^2} = \frac{1 - y^2}{(1 + y)^2} = \frac{1 - y}{1 + y} \\
\end{align*}
\]
consequently
\[
\begin{align*}
  y &= \frac{1 - t^2}{1 + t^2},
  x &= \frac{2t}{1 + t^2}.
\end{align*}
\]
The map \( p = r_1^{-1} : S^1 - \{(0, -1)\} \rightarrow \mathbb{R} \) is well known under the cartographic denomination of “stereographic projection”. It can be explicitly described by the formula
\[
p((x, y)) = \frac{x}{y + 1}
\]
for all \((x, y) \in S^1 - \{(0, -1)\} \) which readily generalises to higher dimensions.

In these lectures we shall study those properties of and artifacts attached to curves which remain invariant under reparametrisations in a sense to become gradually clear in the sequel.

**REMARK 3:** The unit tangent field is invariant under reparametrisations in the following sense: Let \( T_1 : J_1 \rightarrow \mathbb{R}^n \) and \( T_2 : J_2 \rightarrow \mathbb{R}^n \) be the unit tangent fields associated with the curves \( r_1 : J_1 \rightarrow \mathbb{R}^n \) and \( r_2 : J_2 \rightarrow \mathbb{R}^n \) respectively. Suppose that \( r_2 \) is obtained from \( r_1 \) by a reparametrisation \( \rho : J_2 \rightarrow J_1 \). Thus
\[
r_2(\theta) = r_1(\rho(\theta))
\]
and
\[
\dot{r}_2(\theta) = \dot{r}_1(\rho(\theta)) \dot{\rho}(\theta)
\]
and
\[
T_1(\rho(\theta)) = \frac{\dot{\mathbf{t}}_1(\rho(\theta))}{\|\dot{\mathbf{t}}_1(\rho(\theta))\|} = \frac{\dot{\rho}(\theta)\mathbf{t}_1(t)}{\|\dot{\rho}(\theta)\mathbf{t}_1(t)\|} = \frac{\dot{\mathbf{t}}_2(\theta)}{\|\dot{\mathbf{t}}_2(\theta)\|} = T_2(\theta)
\]
since \(\dot{\rho}(\theta) > 0\), for all \(\theta \in J_2\).

**DEFINITION 3**: Given a curve \(\mathbf{r} : J \rightarrow \mathbb{R}^n\) and \(a \in J\), the arclength on \(\mathbf{r}\) measured with initial parameter value \(t = a\) is the function \(s : J \rightarrow \mathbb{R}\) defined by
\[
s(t) = \int_a^t \|\dot{\mathbf{t}}(\tau)\|d\tau.
\]
If \(\mathbf{r}\) is of class \(C^k\), clearly \(s\) is of class \(C^k\), too.

**EXAMPLE 3**: Consider the straight line \(\mathbf{r}(t) = a + tb\), where, of course \(a \in \mathbb{R}^3\), \(b \in \mathbb{R}^3 - \{0\}\). The arclength on \(\mathbf{r}\) measured with initial parameter value \(t = 0\) is
\[
s(t) = \int_0^t \|b\|d\tau = \|b\|t.
\]

**EXAMPLE 4**: Consider circular helix \(\mathbf{r}(t) = (\cos t, \sin t, t)\). The arclength on \(\mathbf{r}\) measured with initial parameter value \(t = 0\) is
\[
s(t) = \int_0^t \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} \, d\tau = \sqrt{2} \, t.
\]

**EXAMPLE 5**: Even very simple curves can give rise to arclengths which are computationally quite intractable: Consider \(\mathbf{r}(t) = (t, t^2)\). The arclength on \(\mathbf{r}\) measured with initial parameter value \(t = 0\) is
\[
s(t) = \int_0^t \sqrt{1 + 4t^2} \, dt = \frac{1}{2} \, t\sqrt{1 + 4t^2} + \frac{1}{4} \ln (2t + \sqrt{1 + 4t^2}).
\]

**REMARK 4**: Given a curve \(\mathbf{r} : J \rightarrow \mathbb{R}^n\) with arclength \(s\) and \(a, b \in J\), the quantity \(|s(b) - s(a)|\) can be interpreted as the length of the segment of the curve \(\mathbf{r}\) traced out as the parameter \(t\) ranges from \(a\) to \(b\). The quantity \(s(b) - s(a)\) may be naturally referred to as the directed length of the same curvilinear segment.

**REMARK 5**: The choice of the initial value of the parameter which has to be introduced while defining the arclength on a curve is immaterial in the sense that arclength functions defined by using different starting values will differ only by a constant: Indeed, if \(s, \tilde{s}\) are arclength functions on \(\mathbf{r}\) measured with the respective initial parameter values \(a, \tilde{a}\), then
\[
\tilde{s}(t) - s(t) = \int_{\tilde{a}}^t \|\dot{\mathbf{t}}(\tau)\|d\tau - \int_a^t \|\dot{\mathbf{t}}(\tau)\|d\tau = \int_{\tilde{a}}^a \|\dot{\mathbf{t}}(\tau)\|d\tau
\]
which is a constant. It should be observed that, this constant equals exactly the directed length of the curvilinear segment of \(\mathbf{r}\) traversed as \(t\) ranges from \(\tilde{a}\) to \(a\) measured by any arclength defined on \(\mathbf{r}\) no matter with which initial value.

**REMARK 6**: Just like the unit tangent field and the tangent lines associated with a curve, the arclength is an artifact which is “geometric”, that is, it is invariant under
reparametrisations : To see this let \( \tau_2 : J_2 \to \mathbb{R}^n \) be a curve which is obtained from \( \tau_1 : J_1 \to \mathbb{R}^n \) by means of a reparametrisation \( \rho : J_2 \to J_1 \). We have \( \tau_2(\theta) = \tau_1(\rho(\theta)) \) for all \( \theta \in J_2 \). Let \( s_2 \) be the arclength on \( \tau_2 \) with initial parameter value \( \theta_0 \in J_2 \). If \( s_1 \) is the arclength on \( \tau_1 \) with initial parameter value \( t_0 = \rho(\theta_0) \in J_1 \), then

\[
s_2(\theta) = \int_{\theta_0}^{\theta} \| \dot{\tau}_2(\tau) \| d\tau = \int_{\theta_0}^{\theta} \| \dot{\tau}_1(\rho(\tau)) \| \rho'(\tau) d\tau = \int_{t_0=\rho(\theta_0)}^{\rho(\theta)} \| \dot{\tau}_1(\sigma) \| d\sigma = s_1(\rho(\theta)).
\]

**DEFINITION 4 :** A curve \( \tau : J \to \mathbb{R}^n \) with arclength \( s : J \to \mathbb{R} \) is said to be **parametrised by arclength** (or a curve of **unit speed**) if \( s(t) - t \) is a constant. \( \tau \) is said to be **parametrised proportionally to arclength** (or to be a **curve of constant speed**) if \( s(t) - \alpha t \) is a constant for some constant \( \alpha > 0 \).

In other words : A curve \( \tau : J \to \mathbb{R}^n \) with arclength \( s : J \to \mathbb{R} \) is said to be parametrised by arclength if \( b-a \) is equal to the directed length of the curvilinear segment of \( \tau \) traversed as \( t \) ranges from \( a \) to \( b \). This statement is of course quite independent of the choice of the initial value chosen for the arclength function.

Similarly \( \tau \) is parametrised proportionally to arclength iff the directed length of the curvilinear segment of \( \tau \) traversed as \( t \) ranges from \( a \) to \( b \) is equal to \( \alpha(b-a) \) for some constant \( \alpha > 0 \).

It should also be noticed that \( \tau \) is a curve of constant speed iff \( \| \dot{\tau}(t) \| \) is constant. Similarly \( \tau \) is parametrised by arclength iff \( \| \dot{\tau}(t) \| = 1 \).

**EXAMPLE 6 :** Clearly the straight line and the circular helix treated in examples 4, 5 are curves of constant speed. It can be checked that

\[
\tau(t) = \left( \frac{5}{13} \cos t, \frac{28}{193} - \sin t, -\frac{12}{13} \cos t \right)
\]

is parametrised by arclength.

**REMARK 7 :** Clearly not every curve is parametrised by arclength. However every curve can be reparametrised to obtain a curve parametrised by arclength. Conversely every curve is seen to be a curve obtained by reparametrisation from a curve parametrised by arclength. Indeed the arclength function \( s \) of a curve \( \tau : J \to \mathbb{R}^n \) satisfies \( \dot{s}(t) = \| \dot{\tau}(t) \| > 0 \) hence \( s : J \to J \) is a reparametrisation for some suitable interval \( J \). Putting \( \tilde{\tau} = \tau \circ s^{-1} : \tilde{J} \to \mathbb{R}^n \) it is found that \( \tau \) is obtained from \( \tilde{\tau} \) by reparametrisation with \( s : J \to \tilde{J} \). The curve \( \tilde{\tau} \) is said to be obtained by **parametrising \( \tau \) by arclength**. The rather unwieldy terminology is best avoided by memorising the optically more efficient

\[
\tau(t) = \tilde{\tau}(s(t)).
\]

Note that \( \tilde{\tau} \) is indeed of unit speed since

\[
\left\| \frac{d\tilde{\tau}}{ds}(s(t)) \right\| = \left\| \frac{1}{s(t)} \frac{d\tilde{\tau}}{dt}(s(t)) \right\| = \left\| \frac{1}{\dot{s}(t)} \frac{d\tau}{dt}(t) \right\| = \frac{1}{\dot{s}(t)} \| \dot{\tau}(t) \| = 1.
\]

**EXAMPLE 7 :** The passage from a given curve to a curve of unit speed as described above is always possible in principle. However, concrete examples in which computations are reasonable are difficult to come by. As an instance consider the catenoid

\[
\tau(t) = (t, \cosh t)
\]
with
\[ s(t) = \int_0^t (1 + \sinh^2 \tau)^{1/2} d\tau = \int_0^t \cosh \tau d\tau = \sinh t \]
and
\[ t = \sinh^{-1} (s) = \log \left( s + \sqrt{s^2 + 1} \right) \]
therefore
\[ \tilde{r}(s) = \left( \log \left( s + \sqrt{s^2 + 1} \right), \sqrt{s^2 + 1} \right) \]

It has been made amply clear that we intend to deal with “geometric” aspects of curves, that is, those aspects which are invariant under reparametrisations. To do this we shall introduce new objects in terms of old objects which are already known to be “geometric”. This will automatically vouchsafe the geometricity of the introduced object. Towards this end we adopt the following useful notation and convention.

**DEFINITION** 5 : Given a curve \( r : J \rightarrow \mathbb{R}^n \) with arclength \( s : J \rightarrow \mathbb{R} \) and a differentiable function \( F : J \rightarrow \mathbb{R}^M \), the arclength rate of change of \( F \) along \( r \) is the function \( F' : J \rightarrow \mathbb{R}^M \) defined by
\[
F'(t) = \lim_{\delta t \to 0} \frac{F(t + \delta t) - F(t)}{s(t + \delta t) - s(t)}
\]
equivalently by
\[
F'(t) = \frac{d(F \circ s^{-1})}{ds}(s(t))
\]
for all \( t \in J \). Higher arclength rates of change can be defined in an inductive fashion by \( F'' = (F')' \), \( F''' = (F'')' = (F')'' \) and quiet generally by \( F^{[k+1]} = (F^{[k]})' = (F')^{[k]} \).

**REMARK** 8 : Since \( F \circ s^{-1}(s(t)) = F(t) \),
\[
\frac{d(F \circ s^{-1})}{ds}(s(t)) \dot{s}(t) = \dot{F}(t)
\]
for all \( t \in J \) by the chain rule and hence
\[
F' = (\dot{s})^{-1} \dot{F}
\]
equivalently
\[
\dot{F} = \dot{s} F'
\]

This shows that obtaining the arclength rate of change of a function \( F \) is tantamount to applying the operator \((\dot{s})^{-1}(d/dt)\) to \( F \).

**REMARK** 9 : Given a curve \( r : J \rightarrow \mathbb{R}^n \) and differentiable functions \( F : J \rightarrow \mathbb{R}^M \), \( A, B : J \rightarrow \mathbb{R}^N \), the reader can easily check the following :
\[
(A \cdot B)' = A' \cdot B + A \cdot B' \quad \text{and} \quad (A \times B)' = A' \times B + A \times B'
\]
\[ \ddot{F} = s^2 F'' + \ddot{s} F', \quad \ddot{F} = s^3 F''' + 3 \dddot{s} F'' + F' \dddot{s} \]

REMARK 10: Arclength rate of change of a function along a curve is invariant in the following sense: Let \( r_1 : J_1 \rightarrow \mathbb{R}^n, r_2 : J_2 \rightarrow \mathbb{R}^n \) be curves with respective arclengths \( s_1 : J_1 \rightarrow \mathbb{R}, s_2 : J_2 \rightarrow \mathbb{R} \). Suppose that \( r_2 : J_2 \rightarrow \mathbb{R}^n \) is obtained from \( r_1 : J_1 \rightarrow \mathbb{R}^n \) by the reparametrisation \( \rho : J_2 \rightarrow J_1 \). Let \( F_1 : J_1 \rightarrow \mathbb{R}^M, F_2 : J_2 \rightarrow \mathbb{R}^M \) be smooth functions related to one another by \( F_2 = F_1 \circ \rho \). We note that \( F'_2 = F'_1 \circ \rho \). Indeed

\[ F'_2(\theta) = (s_2(\theta))^{-1} F_2(\theta) = (s_1(\rho(\theta)) \dot{\rho}(\theta))^{-1} F_1(\rho(\theta)) \dot{\rho}(\theta) = (s_1(\rho(\theta)))^{-1} \dot{F}_1(\rho(\theta)) = F'_1(\rho(\theta)). \]

As an important application of the concept of arclength rate of change, we observe the following very succinct expression for the unit tangent field \( T \) along a curve:

\[ T = \frac{\dot{r}}{\| \dot{r} \|} = (\dot{s})^{-1} \dot{r} = r'. \]

DEFINITION 6: Given a curve \( r : J \rightarrow \mathbb{R}^n \) with arclength \( s : J \rightarrow \mathbb{R} \), the curve \( q : J \rightarrow \mathbb{R}^n \) defined by

\[ q(t) = r(t) - s(t) T(t) \]

for all \( t \in J \), where \( T \) is the unit tangent field of \( r \), is called an involute of \( r \).

The involute of a curve can be visualised as follows: Let a perfectly flexible, inextendable semi-infinite thread be tightly wound onto the curve with its endpoint coinciding with the point with \( s = 0 \) on the curve. The involute is the curve described by the end of the thread as it is unwound without losing its tautness.

It should be noticed that involutes are not uniquely associated with a curve, since their construction depends on the choice the starting value of the parameter in the definition of the arclength.

EXAMPLE 7: It can be readily checked that the involute of the circle

\[ r(t) = (\cos t, \sin t) \]

with \( s(t) = t \) is

\[ r(t) = \left( \cos t + t \sin t, \sin t - t \cos t \right). \]