MATH 371 - DIFFERENTIAL GEOMETRY

FIRST MIDTERM

FAMILY NAME

OTHER NAMES

GRADE


Three questions: \((5 + \sqrt{5}) + 10 + 10 + 5, 10 + 15, 5 + 5 + 15 + 10\)

\(7 + 8\)

\textbf{Solutions}
1. Consider the curve $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3$ defined by

$$\mathbf{r}(t) = \left( t, 1 + \sin t, t - 2 \sin t \right).$$

(A) Compute the unit tangent vector field $\mathbf{T}$ and the binormal field $\mathbf{B}$ along $\mathbf{r}$.
(B) Compute the curvature of $\mathbf{r}$.
(C) Show that $\mathbf{r}$ is a coplanar curve.
(D) Compute the torsion of $\mathbf{r}$.

\[ \dot{\mathbf{v}} = \begin{bmatrix} 1 \\ -\cos t \\ 1 - 2\cos t \end{bmatrix} \]

\[ \overrightarrow{\mathbf{v}} = \frac{4}{\sqrt{1 - 4\cos^2 t + 5 \sin^2 t}} \begin{bmatrix} 1 \\ \cos t \\ 1 - 2\cos t \end{bmatrix} \]

\[ \ddot{\mathbf{v}} = \begin{bmatrix} 0 \\ -\sin t \\ 2\sin t \end{bmatrix} \]

It follows that

\[ \overrightarrow{\mathbf{b}} \text{ has the same direction as } \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{v}} = \begin{bmatrix} \sin t \\ -2\sin t \\ -\sin t \end{bmatrix} \]

\[ \mathbf{b} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \]

\[ K = \frac{\| \overrightarrow{\mathbf{v}} \times \ddot{\mathbf{v}} \|}{\| \ddot{\mathbf{v}} \|^3} = \frac{\sqrt{6} | \sin t |}{(1 - 4\cos^2 t + 5 \cos^3 t)^{3/2}} \]

(C) Clearly $\mathbf{v}'$ lies in the plane $x - 2y - z = -2$

(D) $\tau = 0$ since $\mathbf{v}'$ is coplanar (equivalently $\mathbf{b}$ is constant!)
2. Consider a curve \( \tau : J \rightarrow \mathbb{R}^3 \) of the form \( \tau(t) = (t, \varphi(t), \psi(t)) \) where \( \varphi, \psi : J \rightarrow \mathbb{R} \) are smooth functions.

(A) Prove that \( \tau \) is a regular curve.

(B) Suppose that the second arclength rate of change \( \tau'' \) of \( \tau \) is nowhere vanishing. Prove that in order for the binormal of \( \tau \) at each point to be parallel to the \( yz \)-plane, it is necessary and sufficient that the quantity \( \dot{\varphi} : \dot{\psi} \) is a constant.

\[
(A) \quad \vec{\tau}' = \begin{bmatrix} 1 \\ \dot{\varphi} \\ \dot{\psi} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \forall t \in J, \quad \text{i.e., } \vec{\tau}' \text{ is a regular curve...}
\]

\[
(B) \quad \vec{\tau}'' = \begin{bmatrix} 0 \\ \ddot{\varphi} \\ \ddot{\psi} \end{bmatrix} \quad \text{is independent of } \vec{\tau}', \quad \text{In particular, } \ddot{\varphi}, \ddot{\psi} \text{ cannot simultaneously vanish!}
\]

The binormal \( \vec{b} \) has the same direction as
\[
0 \neq \vec{\tau}' \times \vec{\tau}'' = \begin{bmatrix} \ddot{\varphi} \psi - \ddot{\psi} \varphi \\ * \\ * \end{bmatrix}
\]

\[\therefore \text{In order for } \vec{b} \text{ to be parallel to the } yz\text{-plane, it is necessary and sufficient that } \frac{d}{dt} \left( \frac{\dot{\varphi}}{\dot{\psi}} \right) = \frac{\ddot{\varphi} \psi - \ddot{\psi} \varphi}{\dot{\psi}^2} = 0 \text{ near points with } \dot{\psi} \neq 0 \text{ and } \frac{d}{dt} \left( \frac{\dot{\psi}}{\dot{\varphi}} \right) = 0 \text{ (similarly!) near points with } \dot{\varphi} \neq 0.\]
3. Consider the curve \( r : J \to \mathbb{R}^2 \) defined by

\[
  r(s) = \left( \int_0^s \cos \left[ \int_0^\sigma f(\xi) d\xi \right] d\sigma, \int_0^s \sin \left[ \int_0^\sigma f(\xi) d\xi \right] d\sigma \right)
\]

for each \( s \in J \) where \( f : J \to \mathbb{R} \) is a smooth function.

(A) Prove that \( r \) is parametrised by arclength, in other words, it is a curve of unit speed.

(B) Find the point \( r(0) \). Prove that

\[
  r'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

(C) Prove that the curvature of \( r \) is \( f \).

(D) Find a curve \( q \) in which the curvature \( \kappa \) of the and arclength function \( s \) satisfy

\[
  \kappa = \frac{1}{1 + s^2}
\]

such that \( q(0) = (-2, 7) \) and

\[
  q'(0) = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}
\]

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(A) & (B) **standard. See the lecture notes...**

(c) \[
\kappa = x'' y' - x' y'' = \left[ \cos \left( \int_0^s f(\xi) d\xi \right) \right] \left[ \cos \left( \int_0^s f(\xi) d\xi \right) f(s) \right] - \left[ \sin \left( \int_0^s f(\xi) d\xi \right) f(s) \right] \left[ \sin \left( \int_0^s f(\xi) d\xi \right) f(s) \right] = f(s)
\]

(D) Put \( f(s) = \frac{s}{1 + s^2} \) \( \int_0^s \frac{1}{1 + s^2} d\sigma = \arctan(s) \)

\[
  x(s) = \int_0^s \cos \left( \arctan \sigma \right) d\sigma = \int_0^s \frac{1}{\sqrt{1 + \sigma^2}} d\sigma = \log(\sqrt{1 + s^2} + 1)
\]

\[
  y(s) = \int_0^s \sin \left( \arctan \sigma \right) d\sigma = \int_0^s \frac{\sigma}{\sqrt{1 + \sigma^2}} d\sigma = \sqrt{1 + s^2}
\]

\[
  \sigma = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \end{bmatrix} \text{ where } \begin{bmatrix} x \\ y \end{bmatrix} = r(s)